Linear Realisability over nets and second order quantification

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An overview of Realisability

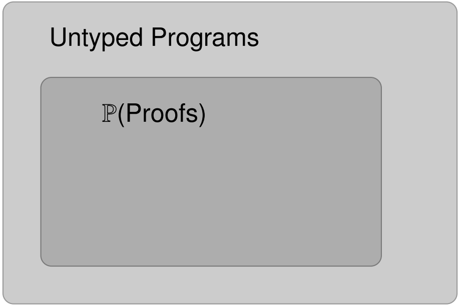
Interactive Framework

Types = class of programs with similar behavior.
A map $[\![\cdot]\!]$: Formulas $\rightarrow \{P \mid P \sim \mathfrak{B}\}$

P : **Nat** $\rightarrow$ **Bool** iff
for any n : **Nat**, $P(n)$: **Bool**

Correctness

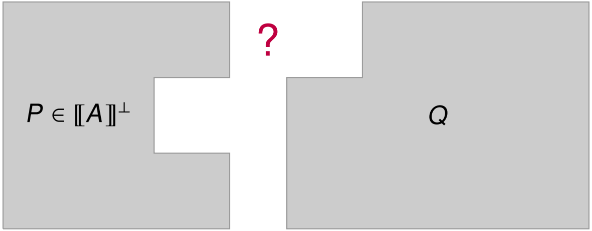
Process $\mathbb{P}$: Proofs $\rightarrow$ Programs.



Types in Orthogonality models (1/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

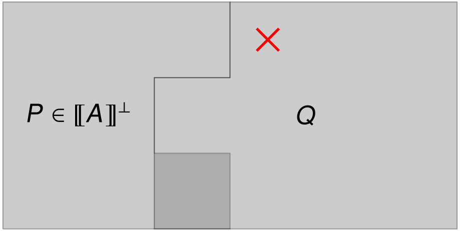
$([\![A]\!] = [\![A]\!]^{\perp\perp})$



Does Q belong to $[\![A]\!]$?

Types in Orthogonality models (2/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$
$([\![A]\!] = [\![A]\!]^{\perp\perp})$

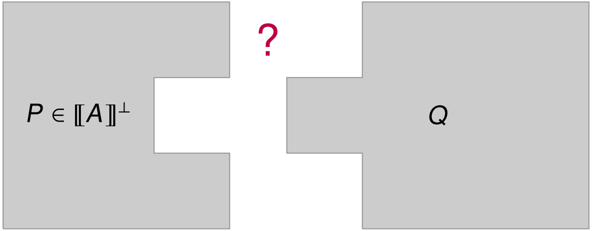


Q fails interaction $\Rightarrow$ $Q \notin [\![A]\!]$

Types in Orthogonality models (3/4)

Realise A = Orthogonal to $[\![A]\!]^\perp$

$([\![A]\!] = [\![A]\!]^{\perp\perp})$

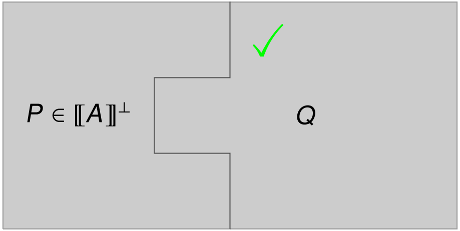


Does Q belong to $[\![A]\!]$?

Types in Orthogonality models (4/4)

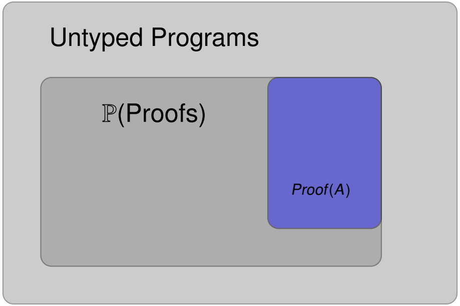
Realise A = Orthogonal to $[\![A]\!]^{\perp}$

$([\![A]\!] = [\![A]\!]^{\perp\perp})$

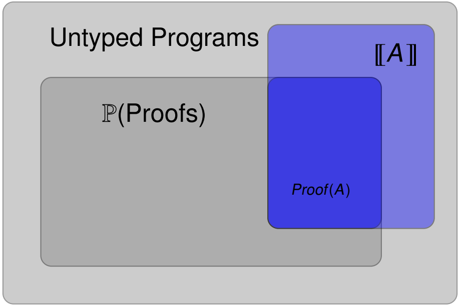


Infer $Q \in [\![A]\!]$.

Adequacy (1/2)



Adequacy (2/2)



Test might not be proofs.

Linear Realisability

Realisability for Linear Logic so far;

- Jean Yves Girard's Geometry of Interaction and Ludics.
- Emmanuelle Beffara's work in process calculi.
- Thomas Seiller Interaction graphs.

No realisability constructions in the standard context of Proof nets yet.

Connections with correctness criterions ?
$S \in \mathbf{A} \Leftrightarrow S \perp \mathbf{A}^{\perp}$.

Realisability Constructions in the space of nets[1]

[1] Here net = proof structure.

Multiplicative nets

A (multiplicative) net is an (hyper)–graph constructed from the following links:

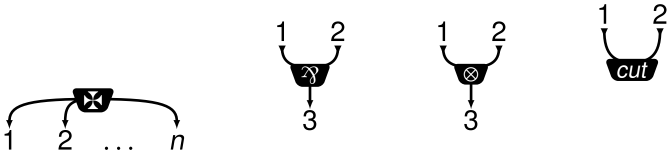


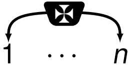
Figure: Links defining the class of multiplicative nets.

Proof nets (1/2)

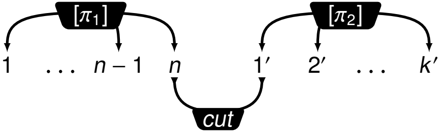
Proofs can be represented by untyped net.

Important information = *order* of the conclusion.

$$\frac{}{\vdash \Gamma}\,\text{✠} =$$

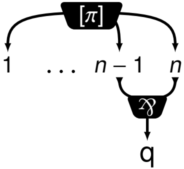


$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\mathrm{cut} =$$

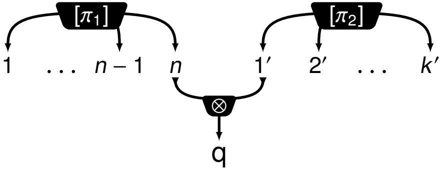


Proof net (2/2)

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \ =$$



$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \ =$$



Correct net = proof net = A net that represents a proof

Some nets are not proof nets

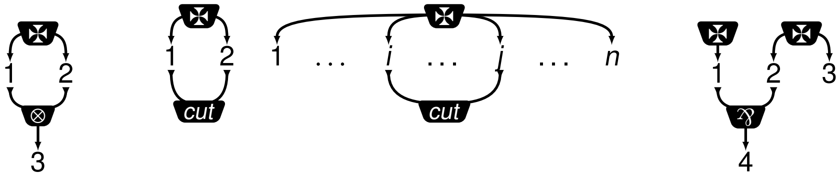
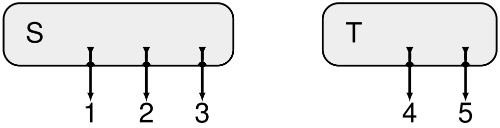


Figure: Examples of multiplicative paraproof structures that do not represent any proof in MLL$^{\maltese}$.

Interaction of nets (1/2)

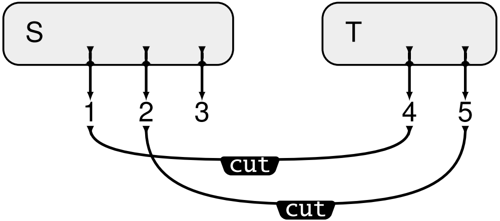
The interaction of two nets

$S \parallel T =$

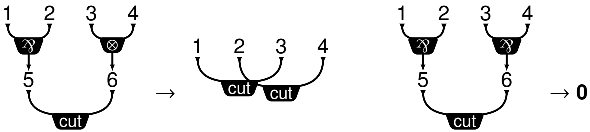


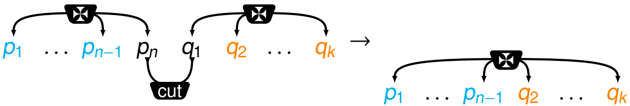
Interaction of nets (2/2)

The interaction of two nets

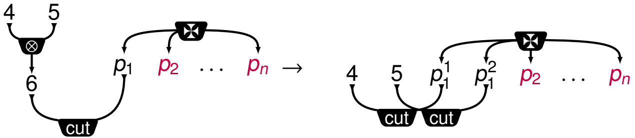
$S :: T \ = \ S \parallel T + \mathtt{cuts} \ =$ 

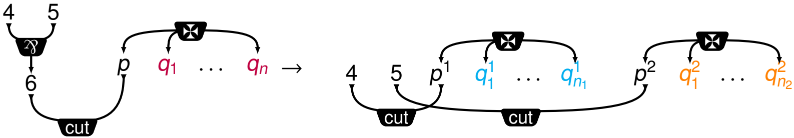
Computation – Homogeneous cut elimination





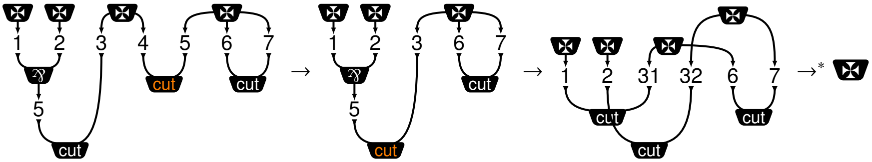
Computation – Non homogeneous cut–elimination



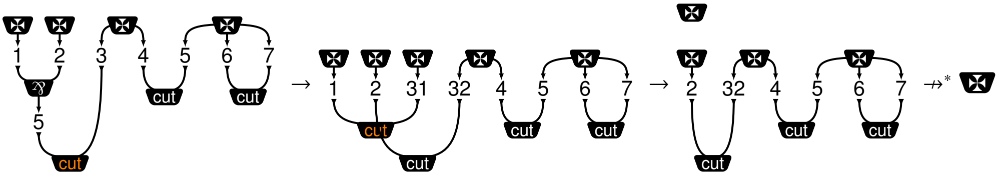


$$\{q_1, \dots, q_n\} = \{q^1_1, \dots, q^1_{n_1}\} \uplus \{q^2_1, \dots, q^2_{n_2}\}$$

Non homogeneous cut–elimination is not confluent (1/2)



Non homogeneous cut–elimination is not confluent (2/2)



Orthogonality for nets

$$S \perp T \Leftrightarrow S :: T \to^* \maltese.$$

For a set A of nets, $A^\perp = \{P \mid \forall a \in A, P \perp a\}$.

Type = Set closed under bi–orthogonality – $\mathbf{A} = \mathbf{A}^{\perp\perp}$.
Equivalently $\mathbf{A} = B^\perp$.

Realisability Constructions (1/2)

Given **A** and **B** two types.

$$\mathbf{A} \circ \mathbf{B} = \{S \mid \forall \overline{a} \in \mathbf{A}^{\perp}, S :: \overline{a} \in \mathbf{B}\}^{\perp\perp}.$$

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp}.$$

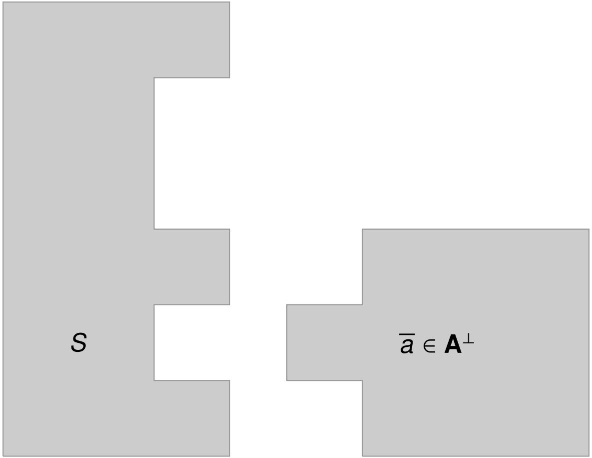
Properties;

- Duality. $(\mathbf{A} \circ \mathbf{B})^{\perp} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$ and $(\mathbf{A} \parallel \mathbf{B})^{\perp} = \mathbf{A}^{\perp} \circ \mathbf{B}^{\perp}$
- Associativity. Both $\circ$ and $\parallel$ are associative.

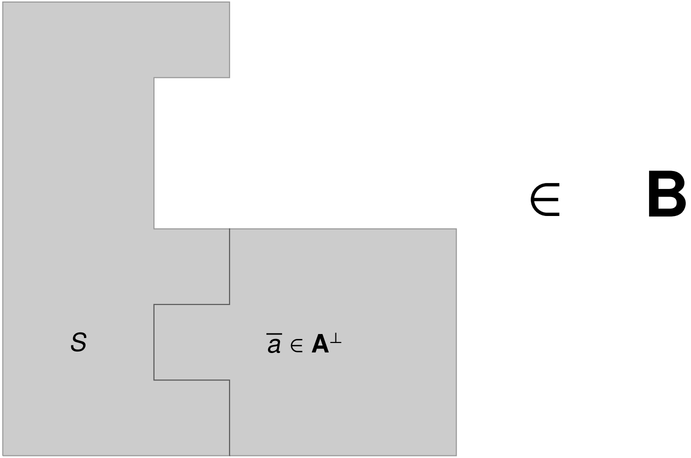
$$(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ (\mathbf{B} \circ \mathbf{C})$$

$$(\mathbf{A} \parallel \mathbf{B}) \parallel \mathbf{C} = \mathbf{A} \parallel (\mathbf{B} \parallel \mathbf{C})$$

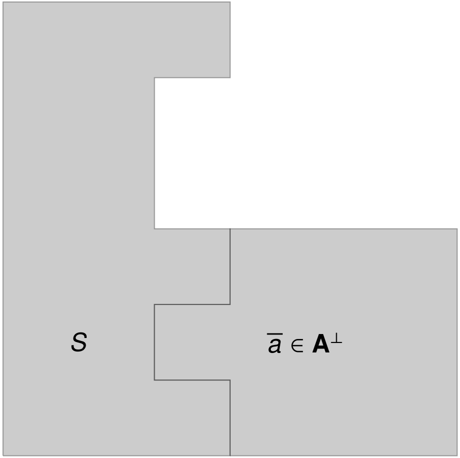
Functional Composition (1/8)



Functional Composition (2/8)

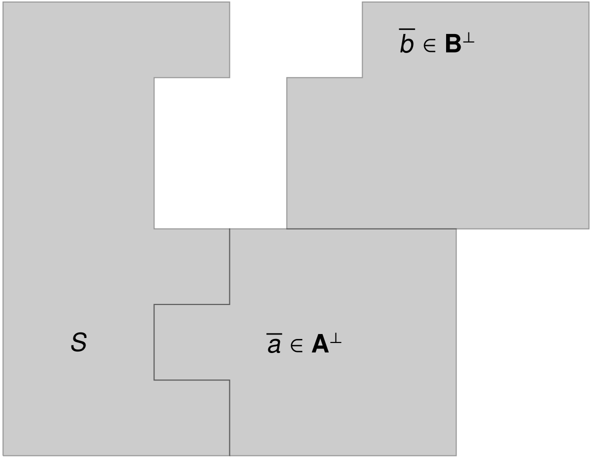


Functional Composition (3/8)

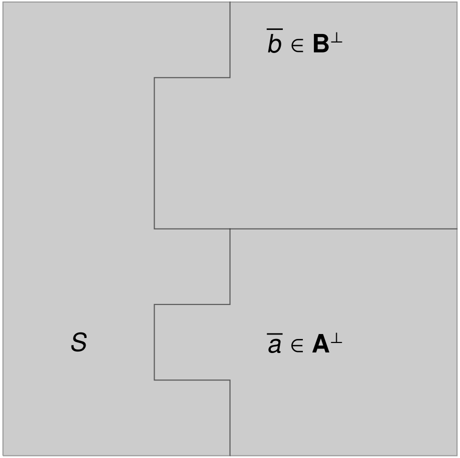


$$\perp \quad \in \quad \mathbf{B}^{\perp}$$

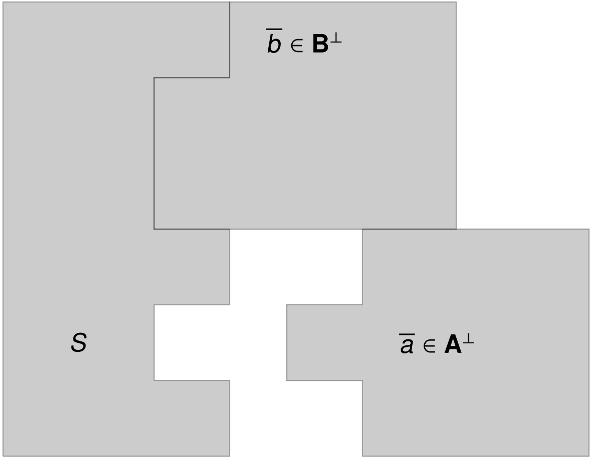
Functional Composition (4/8)



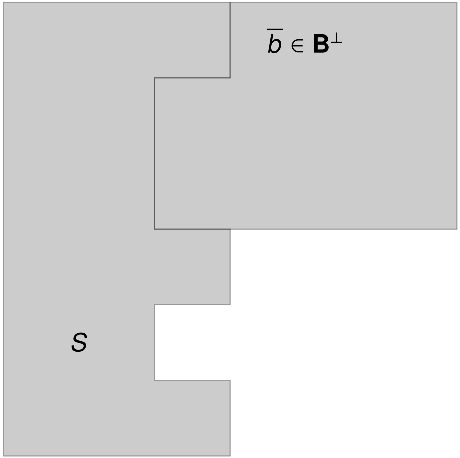
Functional Composition (5/8)



Functional Composition (6/8)

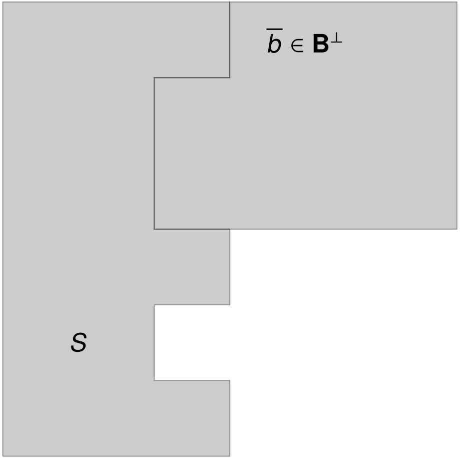


Functional Composition (7/8)



$$\perp \quad \in \quad \mathbf{A}^{\perp}$$

Functional Composition (8/8)



$\in \quad \mathbf{A}$

Realisability Constructions (2/2)

Given $\mathbf{A}$ and $\mathbf{B}$ two types with one conclusion.

$$\mathbf{A} \otimes \mathbf{B} = \{S + \langle S(1), S(2) \rhd_{\otimes} p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\}^{\perp\perp}.$$

$$\mathbf{A} \parr \mathbf{B} = \{S + \langle S(1), S(2) \rhd_{\parr} p \rangle \mid S \in \mathbf{A} \circ \mathbf{B}\}^{\perp\perp}.$$

Properties;

- Duality. $(\mathbf{A} \otimes \mathbf{B})^{\perp} = \mathbf{A}^{\perp} \parr \mathbf{B}^{\perp}$ and $(\mathbf{A} \parr \mathbf{B})^{\perp} = \mathbf{A}^{\perp} \otimes \mathbf{B}^{\perp}$

Interpretation Basis

A map $\mathcal{B} : X \mapsto [\![X]\!]$ such that $[\![X]\!]$ is a type with one conclusion and;

$$[\![X^\perp]\!]_\mathcal{B} = [\![X]\!]_\mathcal{B}^\perp.$$

Lifted to formula's;

$$[\![A \otimes B]\!]_\mathcal{B} = [\![A]\!]_\mathcal{B} \otimes [\![B]\!]_\mathcal{B}.$$

$$[\![A \parr B]\!]_\mathcal{B} = [\![A]\!]_\mathcal{B} \parr [\![B]\!]_\mathcal{B}.$$

Lifted to sequents;

$$[\![A_1, \ldots, A_n]\!]_\mathcal{B} = [\![A_1]\!]_\mathcal{B} \circ \ldots \circ [\![A_n]\!]_\mathcal{B}.$$

Adequacy for MLL and MLL$^{\maltese}$

Adequacy (1/2)

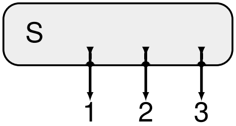
Types contain nets with cuts, in fact;

$$\begin{aligned} S \in [\![A_1]\!]_{\mathcal{B}} \circ \ldots \circ [\![A_n]\!]_{\mathcal{B}} &\Leftrightarrow S \perp [\![A_1]\!]_{\mathcal{B}}^{\perp} \parallel \cdots \parallel [\![A_n]\!]_{\mathcal{B}}^{\perp} \\ &\Leftrightarrow S :: (\overline{a_1} \parallel \cdots \parallel \overline{a_{n-1}}) \perp [\![A_n]\!]_{\mathcal{B}}^{\perp}. \\ &\text{for any } \overline{a_1}, \ldots, \overline{a_{n-1}} \text{ in } [\![A_1]\!]_{\mathcal{B}}^{\perp}, \ldots, [\![A_{n-1}]\!]_{\mathcal{B}}^{\perp}. \end{aligned}$$

Induction cases are easy.

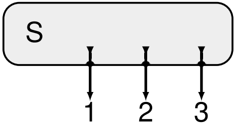
$$\frac{S \in \mathbf{A} \circ \mathbf{B}}{S + \langle S(1), S(2) \rhd_{\parr} p \rangle \in \mathbf{A} \parr \mathbf{B}} \qquad \frac{a \in \mathbf{A} \quad b \in \mathbf{B}}{a \parallel b + \langle a(1), b(1) \rhd_{\parr} p \rangle \in \mathbf{A} \otimes \mathbf{B}}$$

Types contain cuts (1/6)



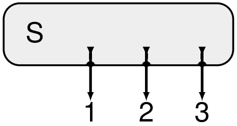
$$\in \quad [\![A_1, A_2, A_3]\!]$$

Types contain cuts (2/6)



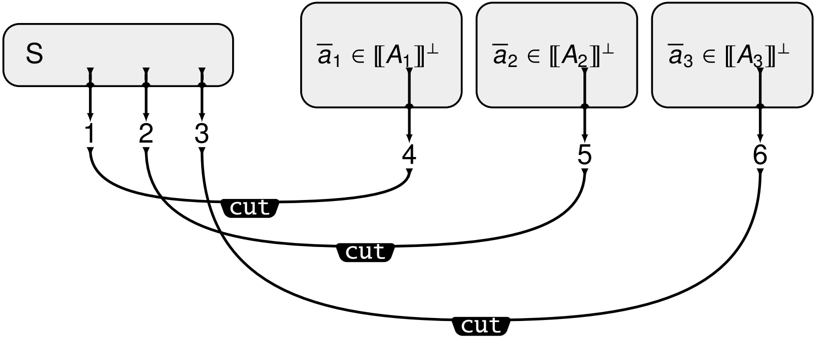
$$\in \quad [\![A_1]\!] \circ [\![A_2]\!] \circ [\![A_3]\!]$$

Types contain cuts (3/6)

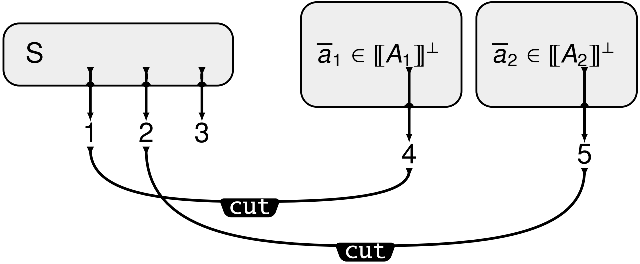


$$\perp \qquad [\![A_1]\!]^{\perp} \parallel [\![A_2]\!]^{\perp} \parallel [\![A_3]\!]^{\perp}$$

Types contain cuts (4/6)

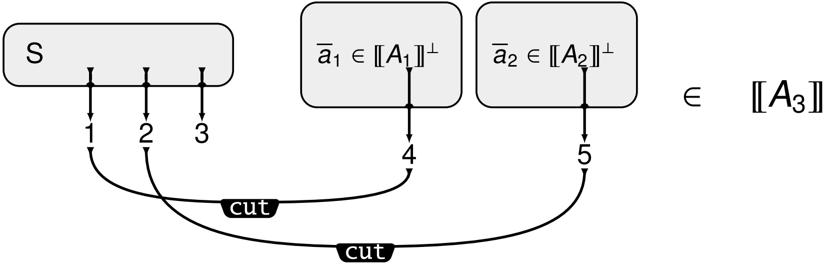


Types contain cuts (5/6)



$\perp \quad [\![A_3]\!]^\perp$

Types contain cuts (6/6)



Adequacy (2/2)

Definition

$\mathcal{B}$ is *approximable* iff for any X, $\maltese_1 \in [\![X]\!]_{\mathcal{B}}$.

Theorem

For any net S and sequent Γ.

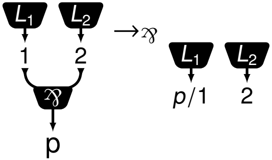
- *For any interpretation basis $\mathcal{B}$,* $\quad S \vdash_{\mathrm{MLL}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$
- *For any approximable interpretation basis $\mathcal{B}$,* $\quad S \vdash_{\mathrm{MLL}^{\maltese}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$

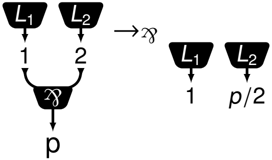
Retrieving proofs: completeness for $\mathrm{MLL}^{\maltese}$

Danos–Regnier correctness criterion (1/2)

Correctness criterion = algorithm that determines if a given net is a proof net.

L_1 and L_2 are two links (potentially the same).





Danos–Regnier correctness criterion (2/2)

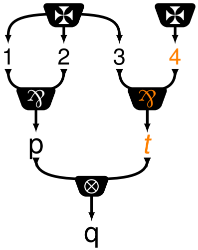
Switching of a net $\mathcal{N}$ = a normal form for the switching reduction.

Theorem (Danos–Regnier, 1989)

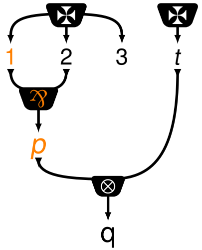
A net $\mathcal{N}$ is correct if and only if all its switchings are connected and acyclic.

Corollary. The switching rewriting preserves correctness.

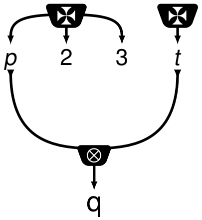
Computing a successful switching (1/3)



Computing a successful switching (2/3)

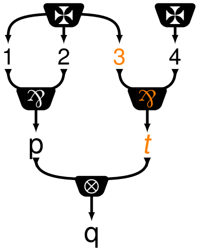


Computing a successful switching (3/3)

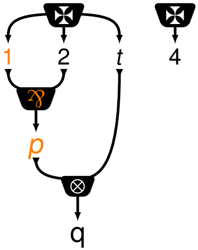


The switching passed the test.

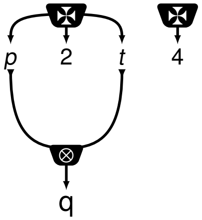
Computing a cyclic switching (1/3)



Computing a cyclic switching (2/3)



Computing a cyclic switching (3/3)



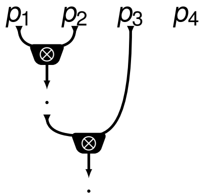
A cycle appeared.

The net has a cyclic switching $\Rightarrow$ NOT a proof net.

DR partitions (1/3)

Switching $\Rightarrow$ partition.
Switching = tensor only net.

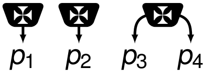
Position belong to the same class iff they belong to the same component.



The partition is $\{\{p_1, p_2, p_3\}, \{p_4\}\}$.

DR partitions (2/3)

Daimon links $\Rightarrow$ partition



The partition is $\{\{p_1\}, \{p_2\}, \{p_3, p_4\}\}$.

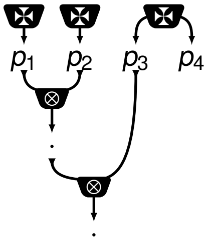
DR partitions (3/3)

Orthogonal partitions $\triangleq$ induced graph is connected+acyclic.

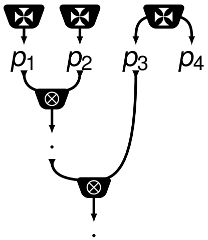
Theorem (Danos–Regnier as partitions)

a net $\mathcal{N}$ is correct iff its upper partition is orthogonal to each partition induced by its switching.

Hence the following is a proof net:

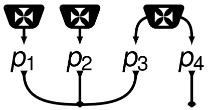


Partitions to Daimons (1/7)



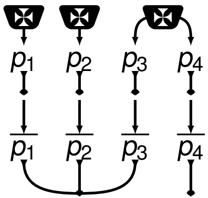
Orthogonal partitions
$\Leftrightarrow$ Acyclic Connected Graph

Partitions to Daimons (2/7)



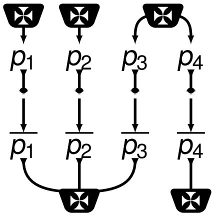
Orthogonal partitions
$\Leftrightarrow$ Acyclic Connected Graph

Partitions to Daimons (3/7)



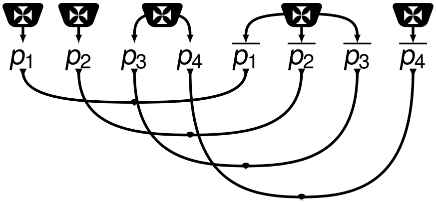
Orthogonal partitions
$\Leftrightarrow$ Acyclic Connected Graph

Partitions to Daimons (4/7)



Orthogonal partitions
$\Leftrightarrow$ Acyclic Connected Graph

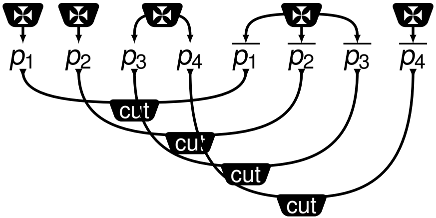
Partitions to Daimons (5/7)



Orthogonal partitions

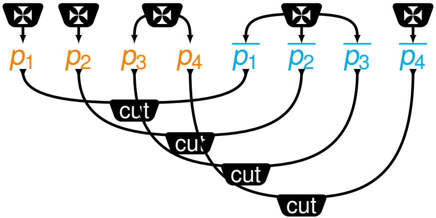
$\Leftrightarrow$ Acyclic Connected Graph

Partitions to Daimons (6/7)



Orthogonal partitions
$\Leftrightarrow$ Acyclic Connected Graph

Partitions to Daimons (7/7)



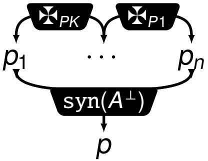
Orthogonal partitions

$\Leftrightarrow$ Acyclic Connected Graph

$\Leftrightarrow$ $S_1 \perp S_2$

Tests are proofs

Theorem. Let A be a formula and $\sigma(A)$ a switching of the syntax tree of A, given $P_1, \ldots, P_K$ are the partitions of the switching $\sigma(A)$;



is correct.

Completeness MLL$^{\maltese}$

Notation. For any X, $[\![X]\!]_{\overline{\mathcal{B}}} = [\![X]\!]_{\mathcal{B}}^{\perp}$.

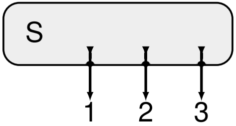
Theorem. For any cut free net S.

$$S \in [\![\Gamma]\!]_{\mathcal{B}} \cap [\![\Gamma]\!]_{\overline{\mathcal{B}}} \Rightarrow S \vdash_{\mathrm{MLL}^{\maltese}} \Gamma.$$

Proof sketch.

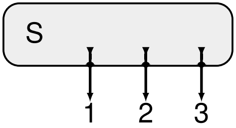
- In this intersection S is 'truncated' and is a sub–forest of Γ.
- S can then interact with the tests for Γ. Using Test=proof + Adequacy

Proof Sketch(1/5)



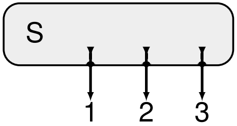
$$\in \quad [\![A_1, A_2, A_3]\!]$$

Proof Sketch (2/5)



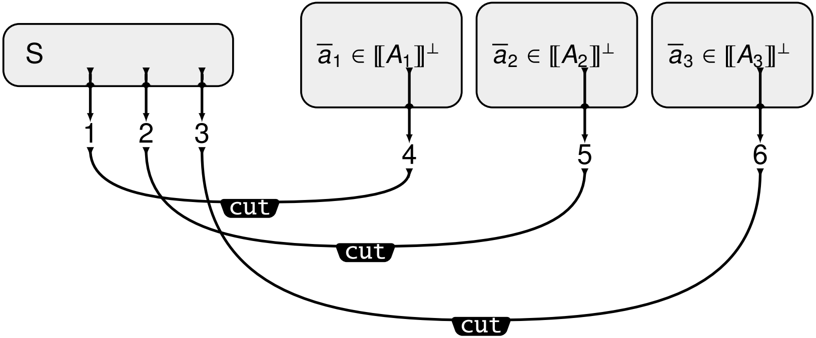
$$\in \quad [\![A_1]\!] \circ [\![A_2]\!] \circ [\![A_3]\!]$$

Proof Sketch (3/5)

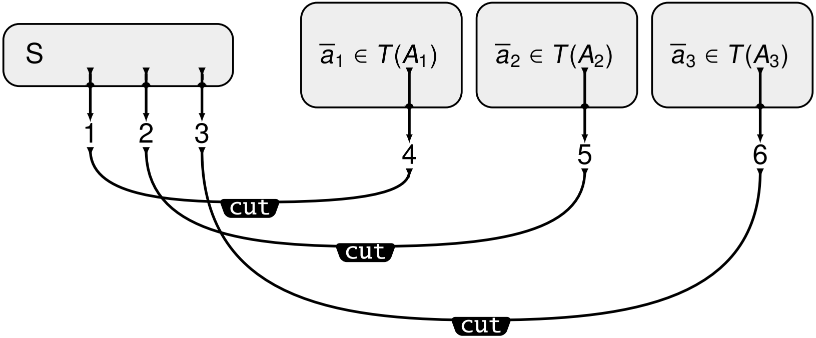


$$\perp \quad [\![A_1]\!]^{\perp} \parallel [\![A_2]\!]^{\perp} \parallel [\![A_3]\!]^{\perp}$$

Proof Sketch (4/5)



Proof Sketch (5/5)



Retrieving proofs: completeness for MLL

Intersection and union type

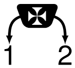
Let Ω be a set of types and $\mathcal{B}$ be an interpretation basis.

$$[\![\bigcap_{X \in \Omega} \Gamma]\!]_{\mathcal{B}} \triangleq \bigcap_{R \in \Omega} [\![\Gamma]\!]_{\mathcal{B}\{X \mapsto R\}}$$

$$[\![\bigcup_{X \in \Omega} \Gamma]\!]_{\mathcal{B}} \triangleq \left(\bigcup_{R \in \Omega} [\![\Gamma]\!]_{\mathcal{B}\{X \mapsto R\}} \right)^{\perp\perp}$$

Completeness MLL

A net is proof like whenever its ✠–link are restricted to:



For a net S proof like and without cuts.

$$S \in \bigcap_{X \in \mathcal{V}} [\![\bigcap_{X \in \Omega} \Gamma]\!]_{\mathcal{B}} \Rightarrow S \vdash_{\text{MLL}} \Gamma.$$

N.B. $\bigcap_{X \in \mathcal{V}} [\![\bigcap_{X \in \Omega} \Gamma]\!]_{\mathcal{B}} = \bigcap_{\mathcal{B}:\text{base}} [\![\Gamma]\!]_{\mathcal{B}}$

Realisability for MLL_2 and adequacy.

Adequacy for MLL_2

Realisability constructions;

$$[\![\forall X A]\!]_{\mathcal{B}} = \{S + \langle S(1) \rhd_{\forall} p\rangle \mid S \in [\![\bigcap_{X \in \Omega} A]\!]_{\mathcal{B}}\}^{\perp\perp}$$

$$[\![\exists X A]\!]_{\mathcal{B}} = \{S + \langle S(1) \rhd_{\exists} p\rangle \mid S \in [\![\bigcup_{X \in \Omega} A]\!]_{\mathcal{B}}\}^{\perp\perp}$$

Theorem (MLL_2 Soundness)

Let $\mathcal{B}$ be an interpretation basis and S be a multiplicative second order net.

$$S \vdash_{\mathrm{MLL}_2} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma.$$

Computability of Types

Computability of Types

A computable $\Leftrightarrow$ exists B finite set s.t. $\mathbf{A} = B^{\perp}$.

The realisability constructions preserve computability;

- $\circ$, $\bigcap_{X \in \Omega}$, $\bigcup_{X \in \Omega}$ preserve computability. Based on the equality $[\![\bigcap_{X \in \Omega} \Gamma]\!]_{\mathcal{B}} = [\![\bigcap_{X \in \{\maltese, \mathtt{T}(\otimes), \mathtt{T}(\parr)\}} \Gamma]\!]_{\mathcal{B}}$
- $\|$ preserve (weak) computability.

These results extend to the sequential constructions $\otimes$, $\parr$, $\forall$, $\exists$.

Conclusion

Conclusion

We provided:

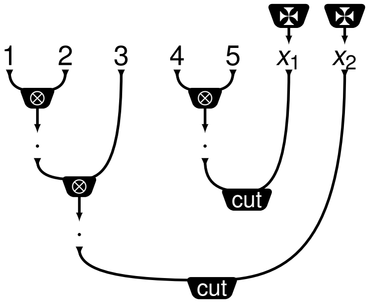
- An adequate realisability model for MLL_2.
- Completeness results for MLL and $MLL^{\maltese}$ based on the 'test–as–proofs' result.
- The orthogonality provides information on the computability of the types; all the constructions preserve computability (or its weak form).

Perspective:

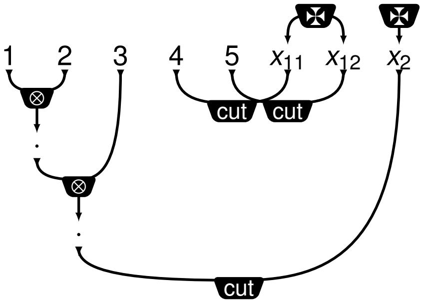
- A new correctness criterion for MLL_2 nets.
- An extension to MLL_2 of the test–as–proofs result.
- Realisabitity for nets of the MELL fragment.

Thank You for Your Attention!

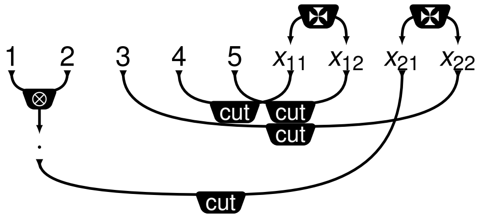
Daimon against ⅋–free net (1/4)



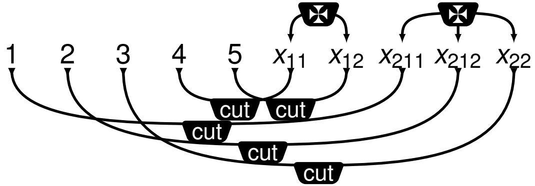
Daimon against ⅋–free net (2/4)



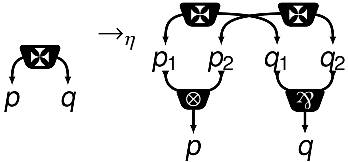
Daimon against ⅋–free net (3/4)



Daimon against ⅋–free net (4/4)



Eta expansion

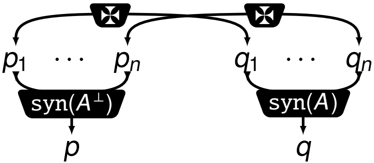


Theorem

The Eta–expansion preserves correctness.

Tests are proofs (1/5)

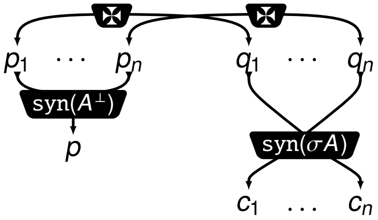
Since $\to_\eta$ preserves correctness:



correct

Tests are proofs (2/5)

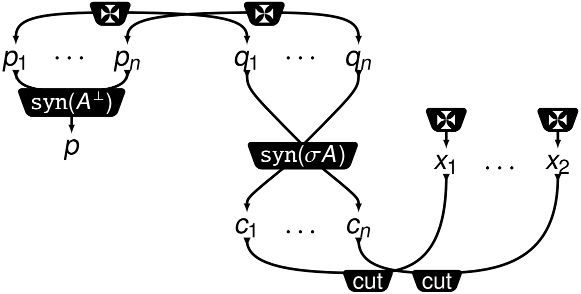
Since $\to_{\parr}$ preserves correctness:



correct

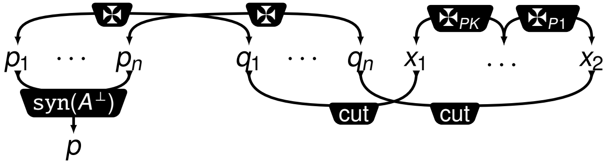
Tests are proofs (3/5)

This preserves correctness:



Tests are proofs (4/5)

Reduction preserves correctness, $P_1, \ldots, P_K$ are the partitions of $\sigma(A)$:



Tests are proofs (5/5)

Reduction preserves correctness, $P_1, \ldots, P_K$ are the partitions of $\sigma(A)$: