# Correctness as Good Interactive Behavior

Master's Thesis

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October 15, 2020

# Logic, intuitions from linguistics

In logic, grossly, we seek to tell if a statement is true or false.

Statements = Formulas
A formula is a syntaxical object, a succesion of symbols.

*Example :*
Socrate is a man, AND,
Socrate is not a man OR Socrate is mortal,
SO, Socrate is mortal.

This statement can be associated to the formula

$$(A \otimes (A^{\perp} \parr B)) \to B$$

Formula = Atomic Formulas + Connectors

# What is a proof?

A proof is an object that allows us to agree on the validity of a formula (under some **context**).

Gentzen's Sequent Calculus = Proof formalism

A one–sided sequent :

$$\vdash A_1, \ldots, A_n$$

It has to be understood as the validity of the formula $A_1 \parr \ldots \parr A_n$

Rules = Premisse to Conclusion
(Intuition) Rules preserve truths. *Example*[1] : Sending $\vdash \Gamma$ and $\vdash \Delta$

to $\vdash \Gamma, \Delta$

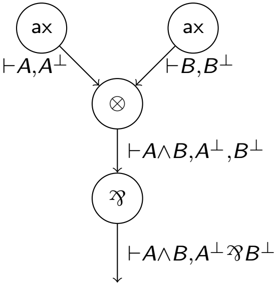
[1] This rule is called the Mix rule

# Proofs as trees

Proofs are then defined as trees such that;

Arrows are labeled by *sequents*.

Nodes are labeled by *rules*.




$$\cfrac{\cfrac{\cfrac{}{\vdash A, A^\perp}\,\mathrm{ax} \qquad \cfrac{}{\vdash B, B^\perp}\,\mathrm{ax}}{\vdash A \otimes B, A^\perp, B^\perp}\,\otimes}{\vdash A \otimes B, A^\perp \parr B^\perp}\,\parr$$

# Multiplicative Linear Logic

$$A, B = X \in \mathcal{V} \quad | \quad A^{\perp} \quad | \quad A \parr B \quad | \quad A \otimes B$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$

$$\frac{}{\vdash A, A^{\perp}}\,\mathsf{Ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,\mathsf{Cut} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes$$

Figure 1: Formulas, de Morgan laws, and rules of the **MLL** fragment

# Multiplicative Proof Structure

A new class of graphs is introduced : **proof structures**.

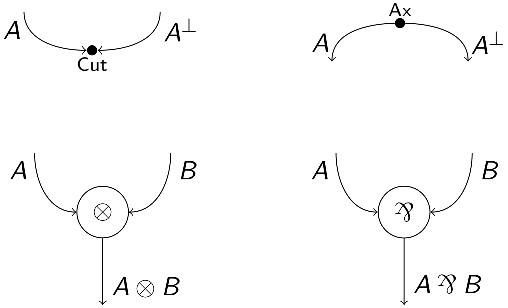



Figure 2: Links for the multiplicative proof structures.

# **MLL** Proof Nets

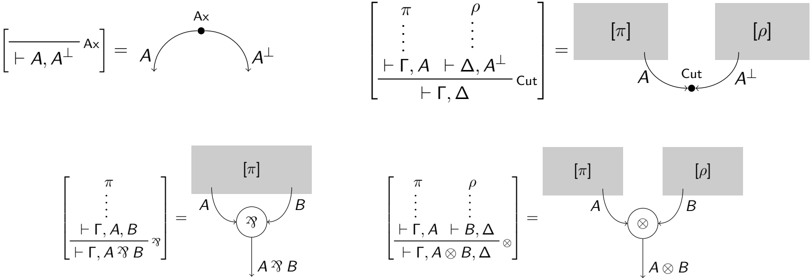



Figure 3: Representation of **MLL** proofs as proof structures

Definition
proof–net = proof structure of the form $[\![\pi]\!]$[2]

[2] ie. there exists a proof $\pi$ of **MLL** such that $S = [\![\pi]\!]$

# Examples, Proof net and Proof Structure

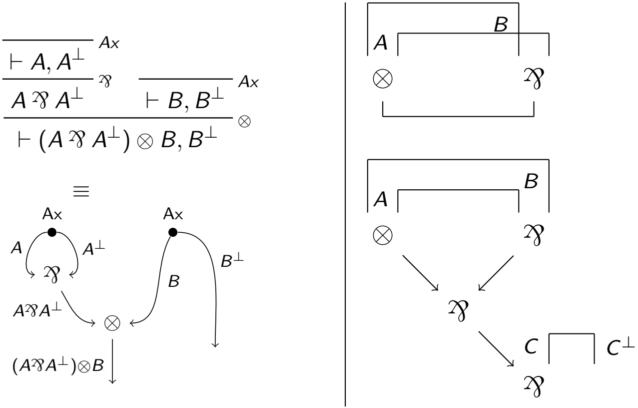



Figure 4: A proof net, and two proof structures

**not all proof structures are proof nets.**

# Correctness, identifying proof nets

not all proof structures are proof nets
$\Rightarrow$ a new problem : *the search for a correctness criterion*.

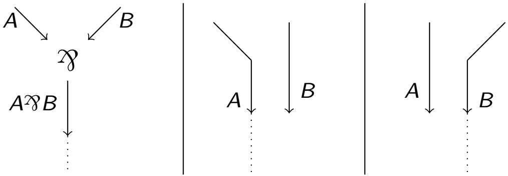



Figure 5: A ⅋ link, And its left and right switching

Definition (Switching)

Swith of a proof structure $S =$ **any** graph obtained after switching each ⅋–node of $S$

Theorem (Correctness Criterion, Danos-Regnier)

*$S$ is a proof net $\Leftrightarrow$ All swiching of $S$ are* ***acyclic*** *and* ***connected****.*

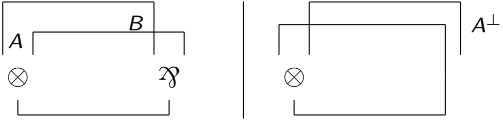



Figure 6: An incorrect proof structure.

# The interaction, cut–elimination

Cut–elimination :

$$\begin{array}{ccc} \pi & & \pi_0 \\ \text{A proof of} \vdash \Gamma & \rightsquigarrow & \text{A proof of} \vdash \Gamma \\ & & \text{without cut.} \end{array}$$

Revealed by the **Curry–Howard correspondence**;
Cut–elimination[3] = Execution of programs[4]

[3] in intuistionistic logic NJ or LJ
[4] Beta–reduction of lambda terms

# Cut Elimination for proof structures

Cut–elimination can also be defined for the class of proof structures.

- proofs = graphs
- cut elimination = graph rewriting

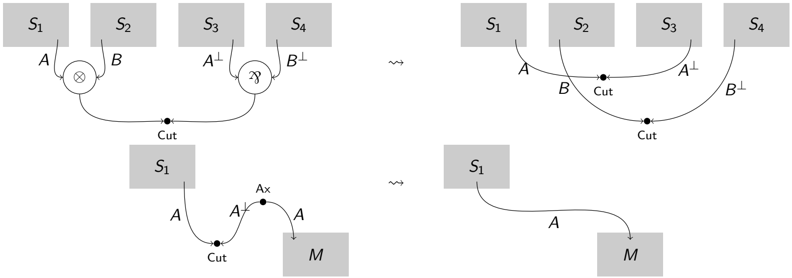



Figure 7: Cut-elimination for multiplicative proof structures.

# Bechet's Work

$$
\begin{tikzcd}
& \text{Proof structures} \arrow[dl, bend right, "\text{its notion of specification}"'] \arrow[dr, bend left, "\text{its notion of calculus}"] & \\
\text{Correctness} \arrow[rr, leftrightarrow, bend right=60, red, "\text{Possible Links?} \to \text{Behavior}"'] & & \text{Cut–elimination}
\end{tikzcd}
$$

# Bad Behavior of proof structures

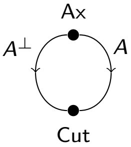



Figure 8: The deadlock proof structure

- basically wrong proof structure = deadlock OR disconnection
- wrong = $S \rightsquigarrow S'$ AND $S'$ basically wrong
- basically bad = $S$ cutted with proofnets $P_1, ..., P_n$ is wrong.
- bad = $\phi(S)$ is basically bad.

# Bechet's Theorem, 1998

Theorem (Bechet)

*For a proof structure* ***without cuts*** *: Incorrectness* $\Leftrightarrow$ *Bad Behavior*

Our focus : *Incorrectness* $\Rightarrow$ *Bad*

Two cases for switching : disconnected and cyclic

# References I

Bechet. Minimality of the correctness criterion. Danos, Regnier. The structure of Multiplicatives.