An Algebraic Structure for Linear Realisabitity

Journées GT Scalp 2023

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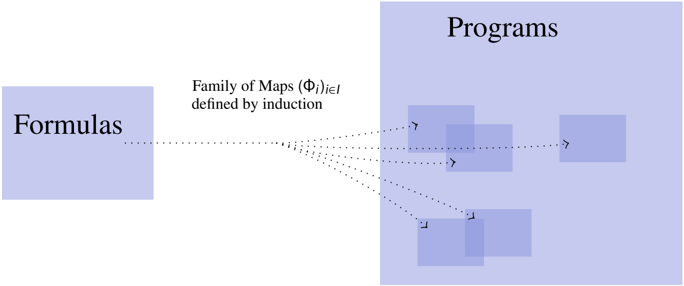
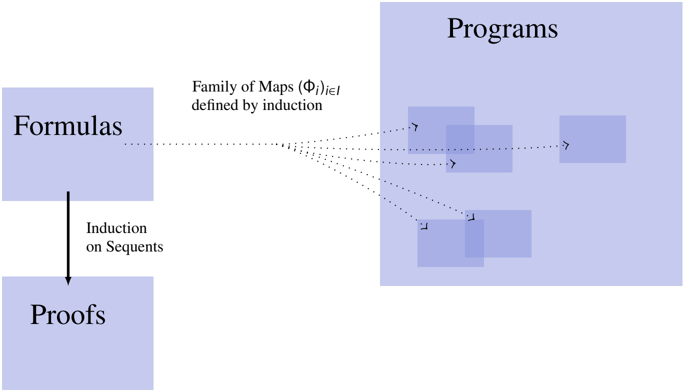
Journées 2023 du GT Scalp

Realisability

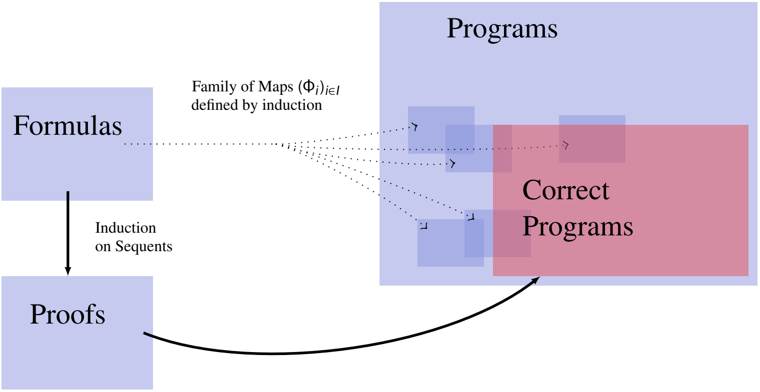
Realisabitity(1/3)



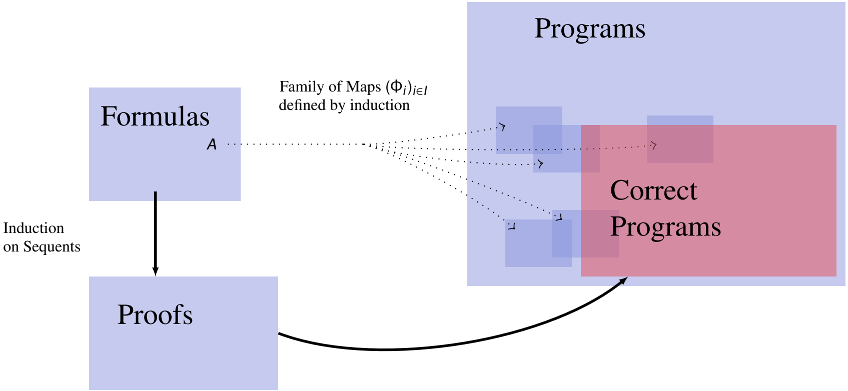
Realisabitity(2/3)



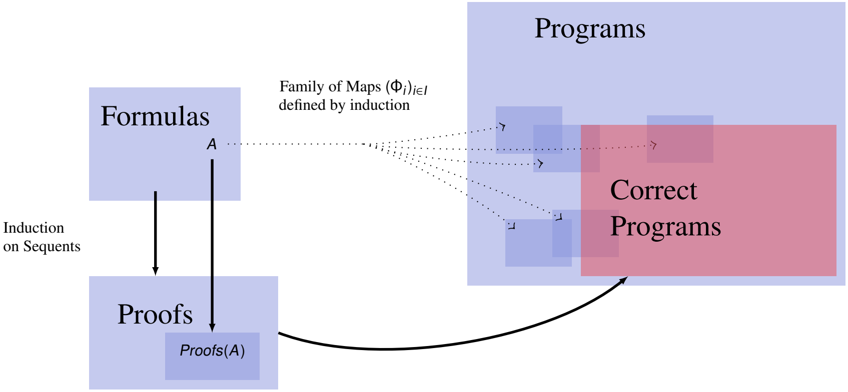
Realisabitity(3/3)



Adequacy(1/4)


Programs
Family of Maps $(\Phi_i)_{i \in I}$
defined by induction
Formulas
A
Correct
Programs
Induction
on Sequents
Proofs

Adequacy(2/4)



Programs
Formulas
A
Family of Maps $(\Phi_i)_{i \in I}$
defined by induction
Correct
Programs
Induction
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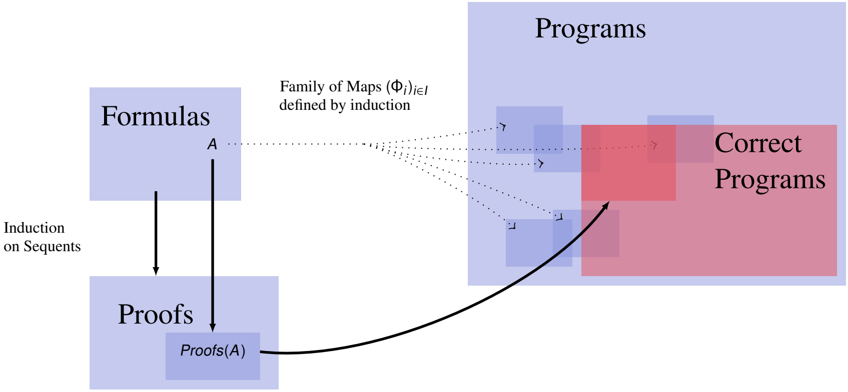
Proofs
$Proofs(A)$

Adequacy(3/4)


Programs
Family of Maps $(\Phi_i)_{i \in I}$
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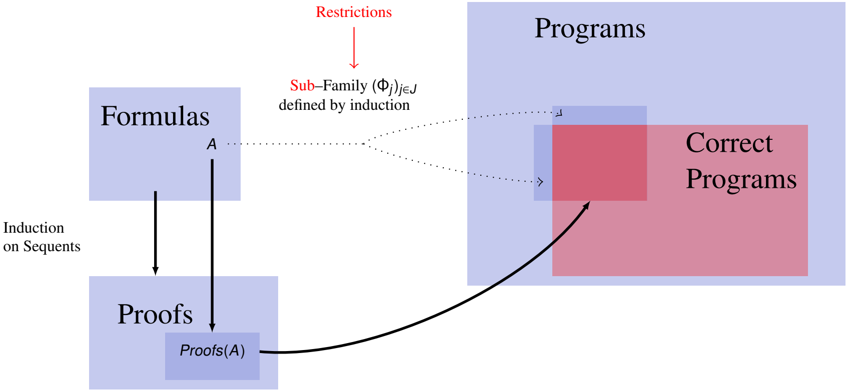
$Proofs(A)$

Adequacy(4/4)


Restrictions
Sub–Family $(\Phi_j)_{j \in J}$
defined by induction
Programs
Formulas
A
Correct
Programs
Induction
on Sequents
Proofs
$Proofs(A)$

Orthogonality

Orthogonality (1/5)

Orthogonality = (symmetric) binary relation $\perp$ on a set X

$A^{\perp} = \{x \in X \mid \forall a \in A, x \perp a\}$.

A is a type $\Leftrightarrow$ $\mathbf{A}^{\perp\perp} = \mathbf{A}$

Orthogonality (2/5)

A polarized magma:

$$(M, \bullet, \perp)$$

A set

Orthogonality (3/5)

A polarized magma:

$$(M, \bullet, \perp)$$

A set

a map $M^2 \to M$

Orthogonality (4/5)

A polarized magma:

$$(M, \bullet, \perp)$$

A set

a subset $\perp \subseteq M$

A map $M^2 \to M$

Orthogonality (5/5)

A polarized magma:

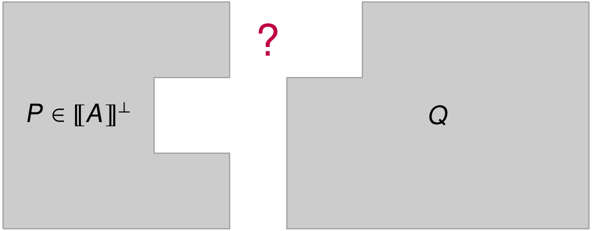
$$(\underbrace{M}_{\text{A set}}, \underbrace{\bullet}_{\text{A map } M^2 \to M}, \underbrace{\perp}_{\text{a subset } \perp \subseteq M})$$

$$a \perp b \Leftrightarrow a \bullet b \in \perp$$

Types in Orthogonality models (1/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

$([\![A]\!] = [\![A]\!]^{\perp\perp})$

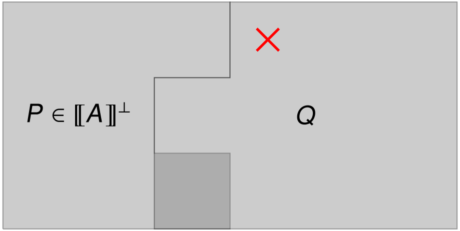


Does Q belong to $[\![A]\!]$?

Types in Orthogonality models (2/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

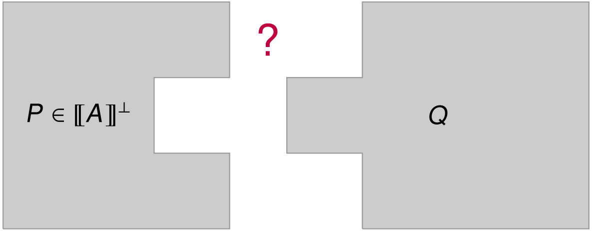
$([\![A]\!] = [\![A]\!]^{\perp\perp})$



Q fails interaction $\Rightarrow$ $Q \notin [\![A]\!]$

Types in Orthogonality models (3/4)

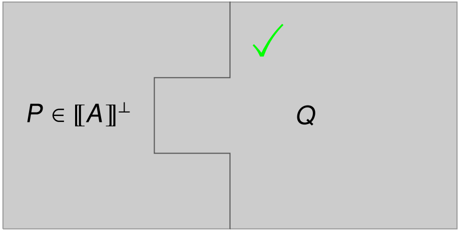
Realise A = Orthogonal to $[\![A]\!]^{\perp}$
$([\![A]\!] = [\![A]\!]^{\perp\perp})$



Does Q belong to $[\![A]\!]$?

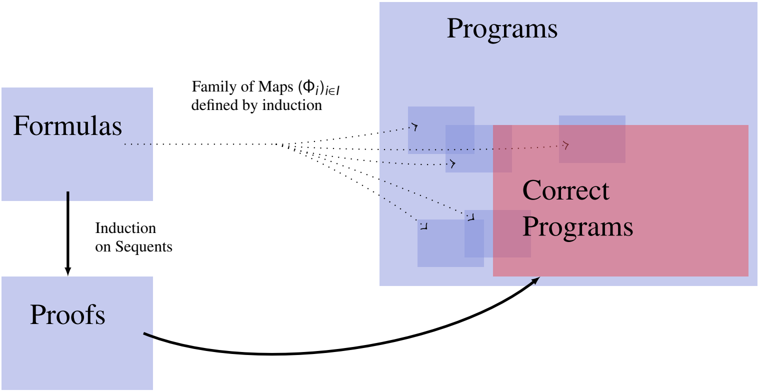
Types in Orthogonality models (4/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$
$([\![A]\!] = [\![A]\!]^{\perp\perp})$

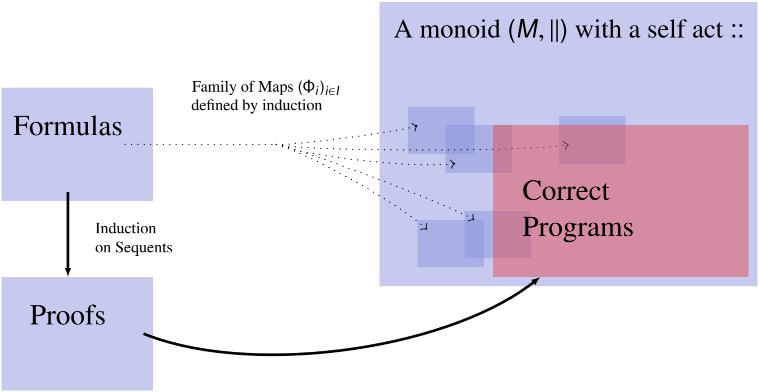


Infer $Q \in [\![A]\!]$.

Realisability in self operand (1/3)

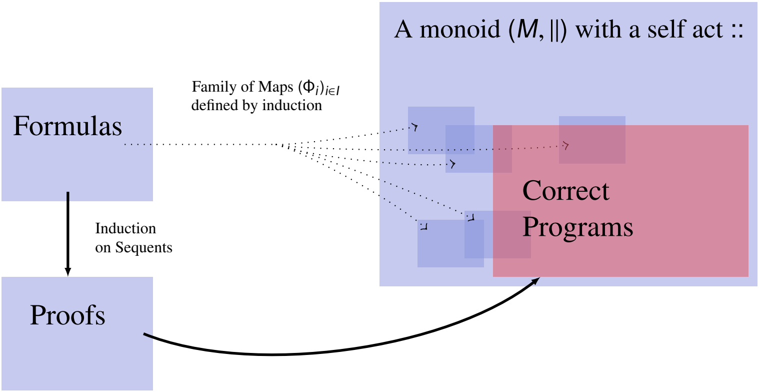


Realisability in self operand (2/3)



$(M, ||)$ acts on the right on M.

Realisability in self operand (3/3)



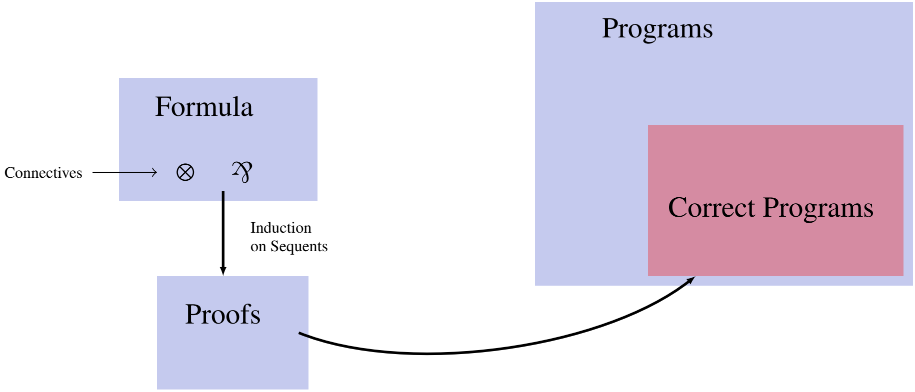
$(M, \|)$ acts on the right on M.

$\forall a, b, c \in M \; a :: (b \parallel c) = (a :: b) :: c.$

Realisability for Multiplicative Linear Logic

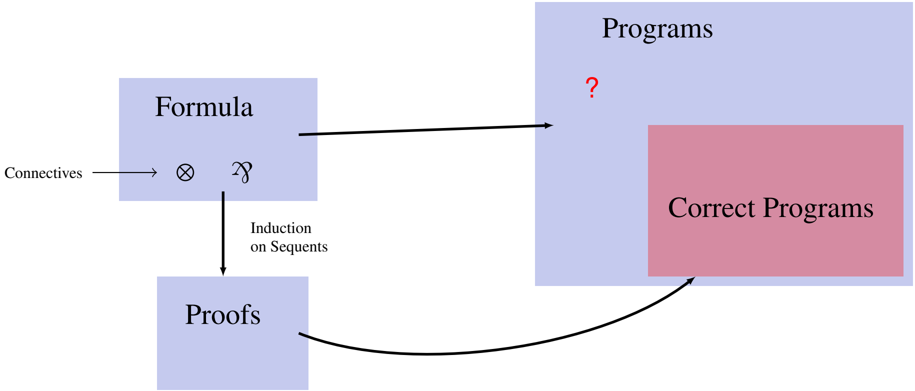
Realisability for MLL (1/4)

(Formula) $\quad A, B \triangleq X, X^{\perp} \mid A \otimes B \mid A \parr B$



Realisability for MLL (2/4)

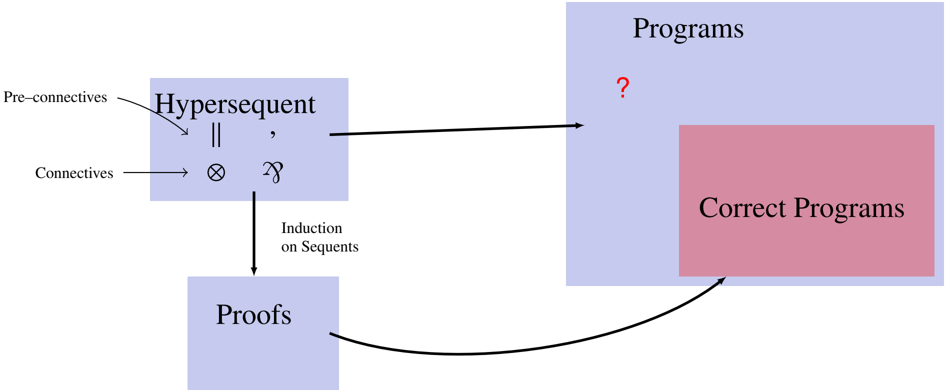
(Formula) $\qquad A, B \triangleq X, X^{\perp} \mid A \otimes B \mid A \parr B$



Realisability for MLL (3/4)

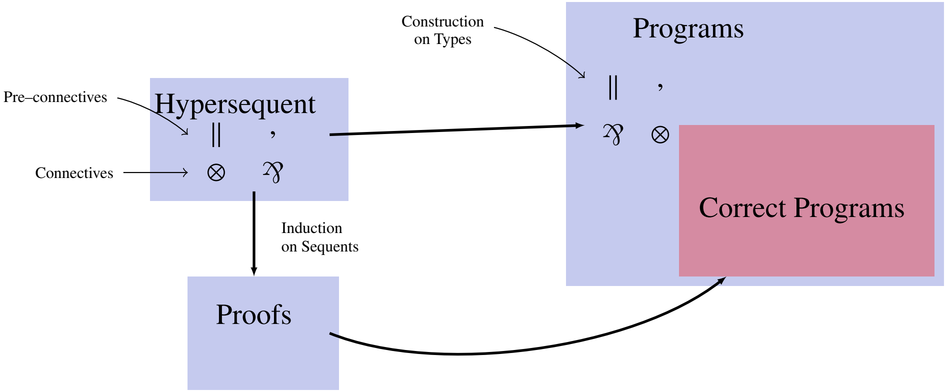
(Formula) $A, B \triangleq X, X^{\perp} \mid A \otimes B \mid A \parr B$

(Hypersequent) $\mathcal{H}_1, \mathcal{H}_2 \triangleq A \mid \mathcal{H}_1, \mathcal{H}_2 \mid \mathcal{H}_1 \parallel \mathcal{H}_2$



Realisability for MLL (4/4)

(Formula)	$A, B \triangleq X, X^\perp \mid A \otimes B \mid A \parr B$
(Hypersequent)	$\mathcal{H}_1, \mathcal{H}_2 \triangleq A \mid \mathcal{H}_1, \mathcal{H}_2 \mid \mathcal{H}_1 \parallel \mathcal{H}_2$



Construction on types in Polarized Self Operand

Construction on Types (1/2)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp}$$

Construction on Types (2/2)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Duality (1/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Duality (2/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch. $x \in \mathbf{A} \circ \mathbf{B} \quad \Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}$

Duality (3/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch.
$$\begin{aligned} x \in \mathbf{A} \circ \mathbf{B} &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}^{\perp\perp} \end{aligned}$$

Duality (4/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch.
$$\begin{aligned} x \in \mathbf{A} \circ \mathbf{B} &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}^{\perp\perp} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \perp \mathbf{B}^{\perp} \end{aligned}$$

Duality (5/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch.
$$\begin{aligned}
x \in \mathbf{A} \circ \mathbf{B} &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B} \\
&\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}^{\perp\perp} \\
&\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \perp \mathbf{B}^{\perp} \\
&\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, \forall \overline{b} \in \mathbf{B}^{\perp}, (x :: \overline{a}) :: \overline{b} \in \perp
\end{aligned}$$

Duality (6/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch.
$$\begin{aligned} x \in \mathbf{A} \circ \mathbf{B} &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}^{\perp\perp} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \perp \mathbf{B}^{\perp} \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, \forall \overline{b} \in \mathbf{B}^{\perp}, (x :: \overline{a}) :: \overline{b} \in \perp \\ &\Leftrightarrow \forall \overline{a} \in \mathbf{A}^{\perp}, \forall \overline{b} \in \mathbf{B}^{\perp}, x :: (\overline{a} \parallel \overline{b}) \in \perp \end{aligned}$$

Duality (7/7)

$$\mathbf{A} \parallel \mathbf{B} = \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \qquad \mathbf{A} \circ \mathbf{B} = \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}^{\perp\perp}$$

Proposition. $\mathbf{A} \circ \mathbf{B} = \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}$.

Proof Sketch.
$$\begin{aligned}
x \in \mathbf{A} \circ \mathbf{B} \quad &\Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B} \\
&\Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \in \mathbf{B}^{\perp\perp} \\
&\Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, x :: \overline{a} \perp \mathbf{B}^{\perp} \\
&\Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, \forall \overline{b} \in \mathbf{B}^{\perp}, (x :: \overline{a}) :: \overline{b} \in \perp \\
&\Leftrightarrow \quad \forall \overline{a} \in \mathbf{A}^{\perp}, \forall \overline{b} \in \mathbf{B}^{\perp}, x :: (\overline{a} \parallel \overline{b}) \in \perp \\
&\Leftrightarrow \quad x \perp \mathbf{A}^{\perp} \parallel \mathbf{B}^{\perp}
\end{aligned}$$

Orthogonality in Self–Operands

Intersections and Unions

Intersection and Union

Given $(\mathbf{A}_i)_{i\in I}$ a family of types:

$\bigcap_{i\in I} A_i$ is a type.

$(\bigcup_{i\in I} A_i)^{\perp} = \left(\bigcup_{i\in I} A_i^{\perp}\right)^{\perp\perp}$.

$(\bigcap_{i\in I} A_i)^{\perp} = \left(\bigcap_{i\in I} A_i^{\perp}\right)^{\perp\perp}$.

Orthogonality in Self–Operands

Operators

Operators and Constructions (1/3)

Operator of arity n on X = map $X^n \to X$.

Higher order operator on X = operator on $\mathcal{P}(X)$

Operator α on $X \Rightarrow$ Higher order operator $\alpha_{\uparrow}$ on X

$$\alpha_{\uparrow} : X_1, \ldots, X_n \to \{\alpha(x_1, \ldots, x_n) \mid x_i \in X_i\}.$$

Operators and Constructions (2/3)

Construction on types = H.O.O. α such that

$$\mathbf{A}_1, \ldots, \mathbf{A}_n \text{ types } \Rightarrow \alpha(\mathbf{A}_1, \ldots, \mathbf{A}_n) \text{ type.}$$

Bi–dual operator *bd* : $A \subseteq X \mapsto A^{\perp\perp}$

Proposition. For any H.O.O. α, $bd \circ \alpha$ is a construction.

Operators and Constructions (3/3)

Two H.O.O. α and β are orthogonal $\Leftrightarrow \alpha(\mathbf{A}_1, \ldots, \mathbf{A}_n)^\perp = \beta(\mathbf{A}_1^\perp, \ldots, \mathbf{A}_n^\perp)$.

Distributive Properties

Whenever α is an operator on X:

$$\bigcup_{\bar{i} \in \prod I_k} \alpha_{\uparrow}(\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_n}) = \alpha_{\uparrow}(\bigcup_{i_1 \in I_1} \mathbf{A}_1, \ldots, \bigcup_{i_n \in I_n} \mathbf{A}_n)$$

$$\bigcap_{\bar{i} \in \prod I_k} \alpha_{\uparrow}(\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_n}) = \alpha_{\uparrow}(\bigcap_{i_1 \in I_1} \mathbf{A}_1, \ldots, \bigcap_{i_n \in I_n} \mathbf{A}_n)$$

Orthogonality in Self–Operands

Implicative Structures in Self Operands

Implicative Structures (1/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \rightarrow)$$

Implicative Structures (2/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \to)$$

Complete meet lattice

Implicative Structures (3/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \to)$$

A map $\mathcal{S}^2 \to \mathcal{S}$

Complete meet lattice

Implicative Structures (4/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \to)$$

A map $\mathcal{S}^2 \to \mathcal{S}$

Complete meet lattice

1. $\forall a_0, a, b \in \mathcal{S} \quad a_0 \preccurlyeq a \quad \Rightarrow \quad a \to b \preccurlyeq a_0 \to b.$

Implicative Structures (5/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \rightarrow)$$

Complete meet lattice

A map $\mathcal{S}^2 \rightarrow \mathcal{S}$

1. $\forall a_0, a, b \in \mathcal{S} \quad a_0 \preccurlyeq a \quad \Rightarrow \quad a \rightarrow b \preccurlyeq a_0 \rightarrow b.$
2. $\forall a_0, a, b \in \mathcal{S} \quad a_0 \preccurlyeq a \quad \Rightarrow \quad a \rightarrow b \preccurlyeq a_0 \rightarrow b.$

Implicative Structures (6/6)

An Implicative Structure:

$$(\mathcal{S}, \preccurlyeq, \rightarrow)$$

A map $\mathcal{S}^2 \rightarrow \mathcal{S}$

Complete meet lattice

1. $\forall a_0, a, b \in \mathcal{S} \quad a_0 \preccurlyeq a \quad \Rightarrow \quad a \rightarrow b \preccurlyeq a_0 \rightarrow b.$
2. $\forall a_0, a, b \in \mathcal{S} \quad a_0 \preccurlyeq a \quad \Rightarrow \quad a \rightarrow b \preccurlyeq a_0 \rightarrow b.$
3. $\forall B \subseteq \mathcal{A} \quad \curlywedge_{b \in B}(a \rightarrow b) = a \rightarrow \curlywedge_{b \in B} b.$

Application in an Implicative Structures

The **application** in an implicative structure:

$$ab \triangleq \bigcurlywedge\{c \in \mathcal{S} \mid a \preccurlyeq b \to c\}.$$

Implicative Structures in Self Operands (1/3)

$$(\mathbb{T}, \subseteq, \to)$$

Implicative Structures in Self Operands (2/3)

$$(\mathbb{T}, \subseteq, \rightarrow)$$

set of types

Closed under $(A_i)_{i \in I} \mapsto (\bigcup_{i \in I} A_i)^{\perp\perp}$

Implicative Structures in Self Operands (3/3)

$$(\mathbb{T}, \subseteq, \rightarrow)$$

set of types

Closed under $(A_i)_{i\in I} \mapsto (\bigcup_{i\in I} A_i)^{\perp\perp}$

construction on types

$$\mathbf{A} \rightarrow \mathbf{B} \triangleq \mathbf{A}^{\perp} \circ \mathbf{B}$$

Application and Arrow Properties (1/5)

$$\mathbf{A} :: \mathbf{B} \triangleq \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

$$\mathbf{A} \longrightarrow \mathbf{B} \triangleq \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}$$

Application and Arrow Properties (2/5)

$$\mathbf{A} :: \mathbf{B} \triangleq \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

$$\mathbf{A} \to \mathbf{B} \triangleq \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}$$

Proposition.

$$\mathbf{A} :: \mathbf{B} = \mathbf{AB} \triangleq \bigcap \{C \mid A \subseteq B \to C\}$$

Application and Arrow Properties (3/5)

$$\mathbf{A} :: \mathbf{B} \triangleq \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

$$\mathbf{A} \longrightarrow \mathbf{B} \triangleq \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}$$

Proposition.

$$\mathbf{A} :: \mathbf{B} = \mathbf{AB} \triangleq \bigcap\{C \mid A \subseteq B \rightarrow C\}$$

$$\mathbf{A} \rightarrow \mathbf{B} = \bigcup\{C \mid C :: A \subseteq B\}$$

Application and Arrow Properties (4/5)

$$\mathbf{A} :: \mathbf{B} \triangleq \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

$$\mathbf{A} \rightarrow \mathbf{B} \triangleq \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}$$

Proposition.

$$\mathbf{A} :: \mathbf{B} = \mathbf{AB} \triangleq \bigcap \{C \mid A \subseteq B \rightarrow C\}$$

$$\mathbf{A} \rightarrow \mathbf{B} = \bigcup \{C \mid C :: A \subseteq B\}$$

$$A^{\perp} = \bigcup \{C \mid A :: C \subseteq \perp\}$$

Application and Arrow Properties (5/5)

$$\mathbf{A} :: \mathbf{B} \triangleq \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

$$\mathbf{A} \rightarrow \mathbf{B} \triangleq \{x \mid \forall a \in \mathbf{A}, x :: a \in \mathbf{B}\}$$

Proposition.

$$\mathbf{A} :: \mathbf{B} = \mathbf{AB} \triangleq \bigcap \{C \mid A \subseteq B \rightarrow C\}$$

$$\mathbf{A} \rightarrow \mathbf{B} = \bigcup \{C \mid C :: A \subseteq B\}$$

$$A^{\perp} = \bigcup \{C \mid A :: C \subseteq \perp\}$$

$$A^{\perp\perp} = \bigcap \{C^{\perp} \mid A :: C \subseteq \perp\}$$

Computability of Types

Computable types (1/3)

A set **A** is a type iff

$$\mathbf{A} = \mathbf{A}^{\perp\perp}$$

Computable types (2/3)

A set **A** is a type iff

$$\mathbf{A} = \mathbf{A}^{\perp\perp}$$
$$\Leftrightarrow \quad \exists B \; \mathbf{A} = B^{\perp}$$

Computable types (3/3)

A set $\mathbf{A}$ is a computable type iff

$$\mathbf{A} = \mathbf{A}^{\perp\perp}$$

$$\Leftrightarrow \quad \exists B \; \mathbf{A} = \underbrace{B}_{\text{finite}}{}^{\perp}$$

Preserving Computability (1/3)

$$\circ, \|, \bigcap_{X\in\Omega}, \bigcup_{X\in\Omega} \text{ preserve computability}$$

PRECONSTRUCTIONS

Preserving Computability (2/3)

weaker form

$\circ, \|, \bigcap_{X\in\Omega}, \bigcup_{X\in\Omega}$ preserve computability

PreConstructions

Preserving Computability (3/3)

weaker form

$\circ, \|, \bigcap_{X \in \Omega}, \bigcup_{X \in \Omega}$ preserve computability

PreConstructions

$$\Downarrow$$

$\parr, \otimes, \forall X, \exists X$ preserve computability

Constructions

Correctness in self–operands

Descriptions (1/6)

How to map proofs to elements of $(X, ::, \|, \perp)$?

Descriptions (2/6)

How to map proofs to elements of $(X, ::, \|, \perp)$?

The notion of **Description**:

Descriptions (3/6)

How to map proofs to elements of $(X, ::, ||, \perp)$?

The notion of **Description**:

$$(\mathcal{A}, \alpha, \beta, \gamma)$$

Descriptions (4/6)

How to map proofs to elements of $(X, ::, \|, \bot)$?

The notion of **Description**:

$$(\mathcal{A}, \alpha, \beta, \gamma)$$

A subset $\mathcal{A} \subseteq X$
of approximations

Descriptions (5/6)

How to map proofs to elements of $(X, ::, \|, \perp)$?

The notion of **Description**:

$$(\mathcal{A}, \alpha, \beta, \gamma)$$

A subset $\mathcal{A} \subseteq X$
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Binary operator $M^2 \to M$

Descriptions (6/6)

How to map proofs to elements of $(X, ::, \|, \perp)$?

The notion of **Description**:

$$(\mathcal{A}, \alpha, \beta, \gamma)$$

A subset $\mathcal{A} \subseteq X$
of approximations

Unary operators $M \to M$

Binary operator $M^2 \to M$

Desequentialization (1/5)

Approximation = a map $\Phi : \mathit{HS}(\mathrm{MLL}) \to \mathcal{A}$.

$$\frac{}{\Gamma} \qquad \mapsto \qquad \Phi(\Gamma)$$

Desequentialization (2/5)

Approximation = a map $\Phi : \mathit{HS}(\mathrm{MLL}) \to \mathcal{A}$.
Desequentialization = lifting an approximation along description $(\mathcal{A}, \alpha, \beta, \gamma)$;

$$\frac{}{\Gamma} \quad \mapsto \quad \Phi(\Gamma)$$

Desequentialization (3/5)

Approximation = a map $\Phi : HS(\mathrm{MLL}) \to \mathcal{A}$.
Desequentialization = lifting an approximation along description $(\mathcal{A}, \alpha, \beta, \gamma)$;

$$\frac{}{\Gamma} \quad \mapsto \quad \Phi(\Gamma)$$

$$\frac{\begin{array}{c} \pi_1 \Big\Vert \\ \Gamma, A, B \end{array}}{\Gamma, A \parr B} \quad \mapsto \quad \beta(\Phi(\pi_1))$$

Desequentialization (4/5)

Approximation = a map $\Phi : HS(\mathrm{MLL}) \to \mathcal{A}$.
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$$\frac{}{\Gamma} \quad \mapsto \quad \Phi(\Gamma)$$

$$\frac{\overset{\pi_1}{\Vert}\atop{\Gamma, A, B}}{\Gamma, A \parr B} \quad \mapsto \quad \beta(\Phi(\pi_1))$$

$$\frac{\overset{\pi_1}{\Vert}\atop{\Gamma, A} \qquad \overset{\pi_2}{\Vert}\atop{\Delta, B}}{\Gamma, \Delta, A \otimes B} \quad \mapsto \quad \alpha(\Phi(\pi_1), \Phi(\pi_2))$$

Desequentialization (5/5)

Approximation = a map $\Phi : HS(\text{MLL}) \to \mathcal{A}$.
Desequentialization = lifting an approximation along description $(\mathcal{A}, \alpha, \beta, \gamma)$;

$$\frac{}{\Gamma} \quad \mapsto \quad \Phi(\Gamma)$$

$$\frac{\begin{array}{c} \pi_1 \Big\| \\ \Gamma, A, B \end{array}}{\Gamma, A \parr B} \quad \mapsto \quad \beta(\Phi(\pi_1))$$

$$\frac{\begin{array}{c} \pi_1 \Big\| \\ \Gamma, A \end{array} \qquad \begin{array}{c} \pi_2 \Big\| \\ \Delta, B \end{array}}{\Gamma, \Delta, A \otimes B} \quad \mapsto \quad \alpha(\Phi(\pi_1), \Phi(\pi_2))$$

$$\frac{\begin{array}{c} \pi_1 \Big\| \\ \Gamma, B, A, \Delta \end{array}}{\Gamma, A, B, \Delta}\, ex \quad \mapsto \quad \gamma(\Phi(\pi_1))$$

Realizers in a self operand

Realizers (1/2)

How are constructed the **interpretations**?

Realizers (2/2)

How are constructed the **interpretations**?
Using **dual binary operators**:

$$(\epsilon, \overline{\epsilon})$$

Interpretation Basis (1/8)

Interpretation basis $\mathscr{B}$ = a map $[\![\cdot]\!]_{\mathscr{B}} : \mathcal{F}_{\mathsf{MLL}} \to \mathsf{type}(X)$.

Interpretation Basis (2/8)

Interpretation basis $\mathcal{B}$ = a map $[\![\cdot]\!]_{\mathcal{B}} : \mathcal{F}_{\mathsf{MLL}} \to \mathrm{type}(\mathrm{X})$.
(Duality condition) $[\![X^{\perp}]\!]_{\mathcal{B}} \subseteq [\![X]\!]_{\mathcal{B}}^{\perp}$.

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$$\Big\Vert$$
Lifting to Hypersequent
$$\Big\Downarrow$$

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$$\Big\Downarrow \quad \text{Using } (\epsilon, \overline{\epsilon})$$

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$$\Vert$$

Lifting to Hypersequent

$$\Downarrow \quad \text{Using } (\epsilon, \overline{\epsilon})$$

$$[\![A \otimes B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\varepsilon\cdot [\![B]\!]_{\mathcal{B}}.$$

Interpretation Basis (6/8)

Interpretation basis $\mathcal{B}$ = a map $[\![\cdot]\!]_{\mathcal{B}} : \mathcal{F}_{\mathsf{MLL}} \to \mathrm{type}(X)$.
(Duality condition) $[\![X^{\perp}]\!]_{\mathcal{B}} \subseteq [\![X]\!]_{\mathcal{B}}^{\perp}$.

$$\Vert$$

Lifting to Hypersequent

$$\Downarrow \quad \text{Using } (\epsilon, \overline{\epsilon})$$

$$[\![A \otimes B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\varepsilon\cdot [\![B]\!]_{\mathcal{B}}.$$

$$[\![A \parr B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\overline{\varepsilon}\cdot [\![B]\!]_{\mathcal{B}}.$$

Interpretation Basis (7/8)

Interpretation basis $\mathcal{B}$ = a map $[\![\cdot]\!]_{\mathcal{B}} : \mathcal{F}_{\mathsf{MLL}} \to \mathrm{type}(X)$.
(Duality condition) $[\![X^{\perp}]\!]_{\mathcal{B}} \subseteq [\![X]\!]_{\mathcal{B}}^{\perp}$.

$$\Downarrow \text{ Lifting to Hypersequent } \Downarrow \text{ Using } (\epsilon, \overline{\epsilon})$$

$[\![A \otimes B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\varepsilon\cdot [\![B]\!]_{\mathcal{B}}$.

$[\![A \parr B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\overline{\varepsilon}\cdot [\![B]\!]_{\mathcal{B}}$.

$[\![A \mid\mid B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \mid\mid [\![B]\!]_{\mathcal{B}}$.

Interpretation Basis (8/8)

Interpretation basis $\mathcal{B}$ = a map $[\![\cdot]\!]_{\mathcal{B}} : \mathcal{F}_{\mathsf{MLL}} \to \text{type(X)}$.
(Duality condition) $[\![X^{\perp}]\!]_{\mathcal{B}} \subseteq [\![X]\!]_{\mathcal{B}}^{\perp}$.

$$\Big\Vert$$

Lifting to Hypersequent

$$\Big\Downarrow \quad \text{Using } (\epsilon, \overline{\epsilon})$$

$[\![A \otimes B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\varepsilon\cdot [\![B]\!]_{\mathcal{B}}$.

$[\![A \parr B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \cdot\overline{\varepsilon}\cdot [\![B]\!]_{\mathcal{B}}$.

$[\![A \mid\mid B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}} \mid\mid [\![B]\!]_{\mathcal{B}}$.

$[\![A, B]\!]_{\mathcal{B}} = [\![A]\!]_{\mathcal{B}}, [\![B]\!]_{\mathcal{B}}$.

Adequacy
Formulating Adequacy

Defining Adequacy

A desequentialization Φ on a description $(\mathcal{A}, \alpha, \beta, \gamma)$ is **adequate with** an interpretation basis $\mathcal{B}$ on a pair $(\varepsilon, \overline{\varepsilon})$ iff:

$$\exists \pi : \Gamma \; x = \Phi(\pi) \Rightarrow x \in [\![\Gamma]\!]_{\mathcal{B}}$$

$$x \vdash_{\Phi} \Gamma \Rightarrow x \vDash_{\mathcal{B}} \Gamma.$$

Adequacy

Sufficient conditions

Distributive description

A description $(\mathcal{A}, \alpha, \beta, \gamma)$ is **distributive**:

- $\forall x, y \;\; \beta(x :: y) = \beta(x) :: y.$
- $\forall x, y, x', y' \;\; \alpha(x :: y, x' :: y') = \alpha(x :: x') :: y \parallel y'.$

Compatible description

A description $(\mathcal{A}, \alpha, \beta, \gamma)$ is **compatible** with a dual pair $(\varepsilon, \overline{\varepsilon})$:

- $\gamma_i(x) \in [\![\Gamma]\!]_{\mathcal{B}} \Rightarrow x \in [\![(i\ i+1)\Gamma]\!]_{\mathcal{B}}$.
- $\beta \circ \circ$ is included in $\overline{\varepsilon}$.
- α is included in ε.

Coherent Approximation

An approximation $\Phi : H_{\mathrm{MLL}} \to \mathscr{A}$ is **coherent** with an interpretation basis $[\![\cdot]\!]_{\mathscr{B}}$:

$$\Phi(\Gamma) \in [\![\Gamma]\!]_{\mathscr{B}}$$

Coherent Desequentialization = Desequentialization from a coherent approximation.

Adequacy

Theorem. Given a distributive description $(\mathcal{A}, \alpha, \beta, \gamma)$ compatible with $(\varepsilon, \overline{\varepsilon})$.
Any coherent desequentialization Φ is adequate with any interpretation basis $\mathcal{B}$.

Completeness

Completeness
Distributive rewriting system

Distributive rewriting systems (1/3)

A distributive rewriting system:

Distributive rewriting systems (2/3)

A distributive rewriting system:

The terms

$$t_1, t_2 = x \in \mathtt{VAR} \;\mid\; t_1 \cdot\alpha\cdot t_2 \;\mid\; t_1 \cdot\beta\cdot t_2 \;\mid\; t_1 + t_2$$

Distributive rewriting systems (3/3)

A distributive rewriting system:

The terms

$$t_1, t_2 = x \in \mathtt{VAR} \mid t_1 \cdot\alpha\cdot t_2 \mid t_1 \cdot\beta\cdot t_2 \mid t_1 + t_2$$

The reduction rule (with closure):

$$a \cdot\alpha\cdot (b \cdot\beta\cdot c) \to (a \cdot\alpha\cdot b) \cdot\beta\cdot c + (a \cdot\alpha\cdot c) \cdot\beta\cdot b.$$

Rewriting properties (1/3)

What about its rewriting properties?

Rewriting properties (2/3)

What about its rewriting properties?

Adding simple equivalence:

$$t[x \leftarrow t_1 + t_2] \equiv t[x \leftarrow t_1] + t[x \leftarrow t_2]$$

Rewriting properties (3/3)

What about its rewriting properties?

Adding simple equivalence:

$$t[x \leftarrow t_1 + t_2] \equiv t[x \leftarrow t_1] + t[x \leftarrow t_2]$$

Distributive rewriting systems are **confluent** and **strongly normalizing**.

NB. Normal forms are stratified: $\sum_{i \in I} \ \ \beta \cdot_{j \in J} \ \ \alpha \cdot_{k \in K} \mathtt{x}(i, j, k)$.

Completeness
Completeness a la Danos–Regnier
Exponential time complexity

Descriptive set of Types (1/5)

α **switches on** β iff

$$\mathbf{A} \cdot\alpha\cdot (\mathbf{B} \cdot\beta\cdot \mathbf{C}) = (\mathbf{A} \cdot\alpha\cdot \mathbf{B}) \cdot\beta\cdot \mathbf{C} \cup (\mathbf{A} \cdot\alpha\cdot \mathbf{C}) \cdot\beta\cdot \mathbf{B}$$

Descriptive set of Types (2/5)

α **switches on** β iff

$$\mathbf{A} \cdot\alpha\cdot (\mathbf{B} \cdot\beta\cdot \mathbf{C}) = (\mathbf{A} \cdot\alpha\cdot \mathbf{B}) \cdot\beta\cdot \mathbf{C} \cup (\mathbf{A} \cdot\alpha\cdot \mathbf{C}) \cdot\beta\cdot \mathbf{B}$$

A set of types $\mathfrak{C}$ is **descriptive** whenever:

Descriptive set of Types (3/5)

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A set of types $\mathfrak{C}$ is **descriptive** whenever:

1. Closed under $\circ$ and $\|$.

Descriptive set of Types (4/5)

α **switches on** β iff

$$\mathbf{A} \cdot\alpha\cdot (\mathbf{B} \cdot\beta\cdot \mathbf{C}) = (\mathbf{A} \cdot\alpha\cdot \mathbf{B}) \cdot\beta\cdot \mathbf{C} \cup (\mathbf{A} \cdot\alpha\cdot \mathbf{C}) \cdot\beta\cdot \mathbf{B}$$

A set of types $\mathfrak{C}$ is **descriptive** whenever:

1. Closed under $\circ$ and $\|$.
2. $\forall \mathbf{A}, \mathbf{B} \in \mathfrak{C}, \quad \mathbf{A} \circ \mathbf{B} = (\mathbf{A} \cdot\alpha_{\uparrow}\cdot \mathbf{B})^{\perp\perp}$.

Descriptive set of Types (5/5)

α **switches on** β iff

$$\mathbf{A} \cdot\alpha\cdot (\mathbf{B} \cdot\beta\cdot \mathbf{C}) = (\mathbf{A} \cdot\alpha\cdot \mathbf{B}) \cdot\beta\cdot \mathbf{C} \cup (\mathbf{A} \cdot\alpha\cdot \mathbf{C}) \cdot\beta\cdot \mathbf{B}$$

A set of types $\mathfrak{C}$ is **descriptive** whenever:

1. Closed under $\circ$ and $||$.
2. $\forall \mathbf{A}, \mathbf{B} \in \mathfrak{C}, \quad \mathbf{A} \circ \mathbf{B} = (\mathbf{A} \cdot\alpha_{\uparrow}\cdot \mathbf{B})^{\perp\perp}$.
3. $\alpha_{\uparrow}$ switches on $||$ in $\mathfrak{C}$.

Semantics of DRS (1/4)

$(\Sigma, \alpha, \beta, +)$ a DRS

$(\mathfrak{C}, \circ, ||, \cup)$

Semantics of DRS (2/4)

$$(\Sigma, \alpha, \beta, +) \text{ a DRS} \xrightarrow{[\![\cdot]\!]_D} (\mathfrak{C}, \circ, ||, \cup)$$

Semantics of DRS (3/4)

$$
\begin{aligned}
[\![t_1 \ \alpha \ t_2]\!]_D &= [\![t_1]\!]_D \circ [\![t_2]\!]_D \\
[\![t_1 \ \beta \ t_2]\!]_D &= [\![t_1]\!]_D \parallel [\![t_2]\!]_D \\
[\![t_1 \ + \ t_2]\!]_D &= [\![t_1]\!]_D \cup [\![t_2]\!]_D
\end{aligned}
$$

Lifted by induction

$$
(\Sigma, \alpha, \beta, +) \text{ a DRS} \xrightarrow{\quad [\![\cdot]\!]_D \quad} (\mathfrak{C}, \circ, ||, \cup)
$$

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Lifted by induction

$$(\Sigma, \alpha, \beta, +) \text{ a DRS} \xrightarrow{[\![\cdot]\!]_D} (\mathfrak{C}, \circ, ||, \cup)$$

$$t \rightarrow t' \quad \Longrightarrow \quad [\![t]\!]_D = [\![t']\!]_D$$

Interpretation Basis and DRS Semantics (1/3)

$$
\begin{aligned}
[\![t_1 \; \alpha \; t_2]\!]_D &= [\![t_1]\!]_D \circ [\![t_2]\!]_D \\
[\![t_1 \; \beta \; t_2]\!]_D &= [\![t_1]\!]_D \parallel [\![t_2]\!]_D \\
[\![t_1 \; + \; t_2]\!]_D &= [\![t_1]\!]_D \cup [\![t_2]\!]_D
\end{aligned}
$$

Lifted by induction

$$
\begin{tikzcd}
(\mathtt{VAR}, \mathfrak{X}, \otimes, \cup) \text{ a DRS} \arrow[r, "{[\![\cdot]\!]_D}"] & (\mathfrak{C}, \circ, \parallel, \cup)
\end{tikzcd}
$$

$$
t \rightarrow t' \quad \Longrightarrow \quad [\![t]\!]_D = [\![t']\!]_D
$$

Interpretation Basis and DRS Semantics (2/3)

$$\begin{aligned}
[\![t_1 \parr t_2]\!]_D &= [\![t_1]\!]_D \circ [\![t_2]\!]_D \\
[\![t_1 \otimes t_2]\!]_D &= [\![t_1]\!]_D \mid\mid [\![t_2]\!]_D \\
[\![t_1 \cup t_2]\!]_D &= [\![t_1]\!]_D \cup [\![t_2]\!]_D
\end{aligned}$$

Lifted by induction

$$(\mathtt{VAR}, \parr, \otimes, \cup) \text{ a DRS} \xrightarrow{[\![\cdot]\!]_D} (\mathfrak{C}, \circ, \mid\mid, \cup)$$

$$t \to t' \quad \Longrightarrow \quad [\![t]\!]_D = [\![t']\!]_D$$

Interpretation Basis and DRS Semantics (3/3)

$$[\![t_1 \parr t_2]\!]_D = [\![t_1]\!]_D \circ [\![t_2]\!]_D$$
$$[\![t_1 \otimes t_2]\!]_D = [\![t_1]\!]_D \mid\mid [\![t_2]\!]_D$$
$$[\![t_1 \cup t_2]\!]_D = [\![t_1]\!]_D \cup [\![t_2]\!]_D$$

Lifted by induction

$(\mathtt{VAR}, \parr, \otimes, \cup)$ a DRS $\xrightarrow{[\![\cdot]\!]_D}$ $(\mathfrak{C}, \circ, ||, \cup)$

Interpretation Basis

$$t \rightarrow t' \quad \Longrightarrow \quad [\![t]\!]_D = [\![t']\!]_D$$

Danos Regnier Tests (1/4)

$[\![\cdot]\!]_{\mathcal{B}}$ is a denotational semantic.

Danos Regnier Tests (2/4)

$[\![\cdot]\!]_{\mathcal{B}}$ is a denotational semantic.

$$
\begin{aligned}
& A \longrightarrow \bigcup_{i\in I} \circ_{j\in J} \ ||_{k\in k} \ \mathrm{a}(i,j,k) \\
\Rightarrow \quad & [\![A]\!]_{\mathcal{B}} = [\![\bigcup_{i\in I} \circ_{j\in J} \ ||_{k\in k} \ \mathrm{a}(i,j,k)]\!]_{\mathcal{B}}
\end{aligned}
$$

Danos Regnier Tests (3/4)

$[\![\cdot]\!]_{\mathcal{B}}$ is a denotational semantic.

$$
\begin{array}{rl}
& A \longrightarrow \bigcup_{i \in I} \ \circ_{j \in J} \ ||_{k \in k} \ \mathsf{a}(i, j, k) \\
\Rightarrow & [\![A]\!]_{\mathcal{B}} = [\![\bigcup_{i \in I} \ \circ_{j \in J} \ ||_{k \in k} \ \mathsf{a}(i, j, k)]\!]_{\mathcal{B}} \\
\Leftrightarrow & [\![A]\!]_{\mathcal{B}} = \left(\bigcup_{i \in I} \ \circ_{j \in J} \ ||_{k \in k} \ [\![\mathsf{a}(i, j, k)]\!]_{\mathcal{B}}\right)^{\perp\perp}
\end{array}
$$

Danos Regnier Tests (4/4)

$[\![\cdot]\!]_{\mathcal{B}}$ is a denotational semantic.

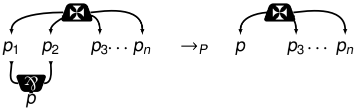
$$
\begin{aligned}
& A \longrightarrow \bigcup_{i \in I} \circ_{j \in J} \|_{k \in k} \ \mathrm{a}(i, j, k) \\
\Rightarrow \quad & [\![A]\!]_{\mathcal{B}} = [\![\bigcup_{i \in I} \circ_{j \in J} \|_{k \in k} \ \mathrm{a}(i, j, k)]\!]_{\mathcal{B}} \\
\Leftrightarrow \quad & [\![A]\!]_{\mathcal{B}} = \left(\bigcup_{i \in I} \circ_{j \in J} \|_{k \in k} \ [\![\mathrm{a}(i, j, k)]\!]_{\mathcal{B}}\right)^{\perp\perp} \\
\Leftrightarrow \quad & [\![A]\!]_{\mathcal{B}}^{\perp} = \left(\bigcap_{i \in I} \|_{j \in J} \circ_{k \in k} [\![\mathrm{a}(i, j, k)]\!]_{\mathcal{B}}\right)^{\perp}
\end{aligned}
$$

Completeness

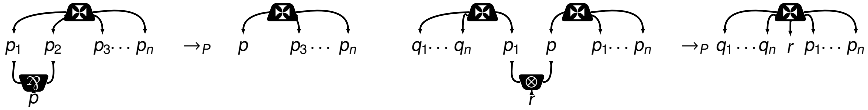
Parsing

Naively $O(n^2)$ time complexity

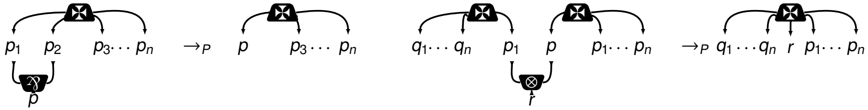
Parsing (1/3)



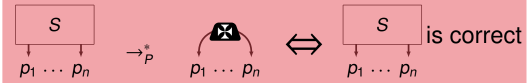
Parsing (2/3)


p_1 p_2 $p_3 \cdots p_n$
p
$\to_P$
p $p_3 \cdots p_n$
$q_1 \cdots q_n$ p_1 p $p_1 \cdots p_n$
r
$\to_P$
$q_1 \cdots q_n$ r $p_1 \cdots p_n$

Parsing (3/3)



Contractibility Criterion



Implementing Parsing (1/2)

$$
\begin{array}{lcl}
(X, \|, ::, \bot) & = & \mathtt{Induction}\{ \\
 & & \quad \mathtt{BASESET} = \mathtt{X}_{\emptyset} \\
 & & \quad \mathtt{StableUnder}\{\|, \alpha, \beta\} \\
 & & \}
\end{array}
$$

Implementing Parsing (2/2)

$$
\begin{array}{rcl}
(X, \|, ::, \bot) & = & \mathtt{Induction}\{ \\
& & \quad \mathtt{BASESET} = \mathtt{X}_{\emptyset} \\
& & \quad \mathtt{StableUnder}\{\|, \alpha, \beta\} \\
& & \}
\end{array}
$$

Parsing = $\alpha(\mathtt{x}) \to \mathtt{z}$ and $\beta(\mathtt{x}, \mathtt{y}) \to \mathtt{z}$.

Completeness

Parsing and orthogonality

$O(n)$ time complexity

Linear Time test (1/7)

Linear time tests in orthogonality models?

Linear Time test (2/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

Linear Time test (3/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

Ingredients:

Linear Time test (4/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

Ingredients:

- Decomposition $x \in X \mapsto \mathtt{x}_1 \parallel \cdots \parallel \mathtt{x}_n$

Linear Time test (5/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

Ingredients:

Decomposition $x \in X \mapsto \mathtt{x}_1 \parallel \cdots \parallel \mathtt{x}_n$

$\alpha(\mathtt{x}) \to_P \mathtt{z} \quad \Leftrightarrow \quad \mathtt{x} :: \mathrm{P}_\alpha \in \perp$

Linear Time test (6/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

Ingredients:

Decomposition $x \in X \mapsto \mathtt{x}_1 \parallel \cdots \parallel \mathtt{x}_n$

$\alpha(\mathtt{x}) \rightarrow_P \mathtt{z} \quad \Leftrightarrow \quad \mathtt{x} :: \mathrm{P}_\alpha \in \perp$

$\beta(\mathtt{x} \parallel \mathtt{y}) \rightarrow_P \mathtt{z} \quad \Leftrightarrow \quad \mathtt{x} \parallel \mathtt{y} :: \mathrm{P}_\beta \in \perp$

Linear Time test (7/7)

Linear time tests in orthogonality models?

Based on unification, parsing has **linear**–complexity. (Guerrini 2011).

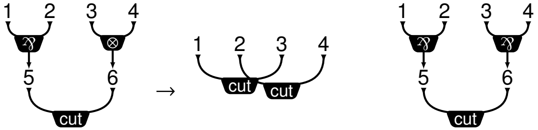
Ingredients:

- Decomposition $x \in X \mapsto \mathtt{x}_1 \parallel \cdots \parallel \mathtt{x}_n$
- $\alpha(\mathtt{x}) \to_P \mathtt{z} \quad \Leftrightarrow \quad \mathtt{x} :: \mathtt{P}_\alpha \in \perp$
- $\beta(\mathtt{x} \parallel \mathtt{y}) \to_P \mathtt{z} \quad \Leftrightarrow \quad \mathtt{x} \parallel \mathtt{y} :: \mathtt{P}_\beta \in \perp$
- Composition for $\mathtt{P}_\alpha$ and $\mathtt{P}_\beta$.

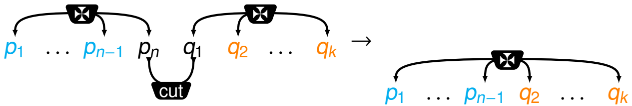
Completeness

Linear Time Tests with nets

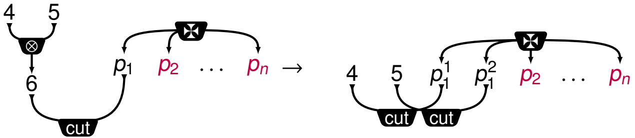
Computation – Homogeneous cut elimination

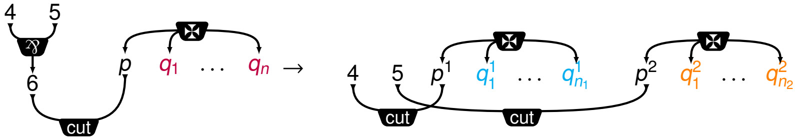


does not reduce



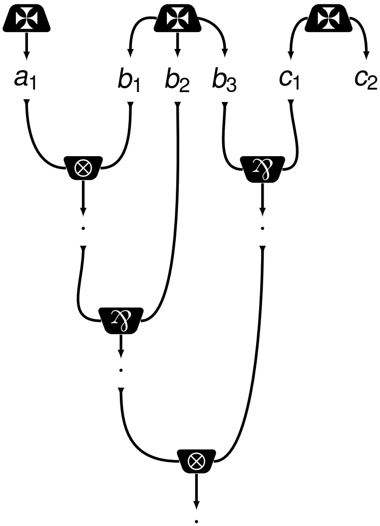
Computation – Non homogeneous cut–elimination



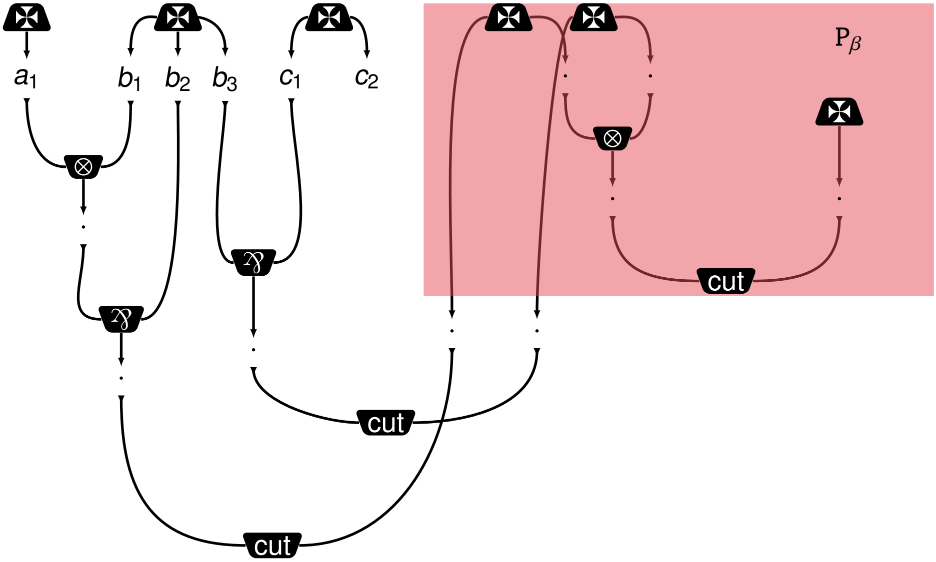


$$\{q_1, \ldots, q_n\} = \{q^1_1, \ldots, q^1_{n_1}\} \uplus \{q^2_1, \ldots, q^2_{n_2}\}$$

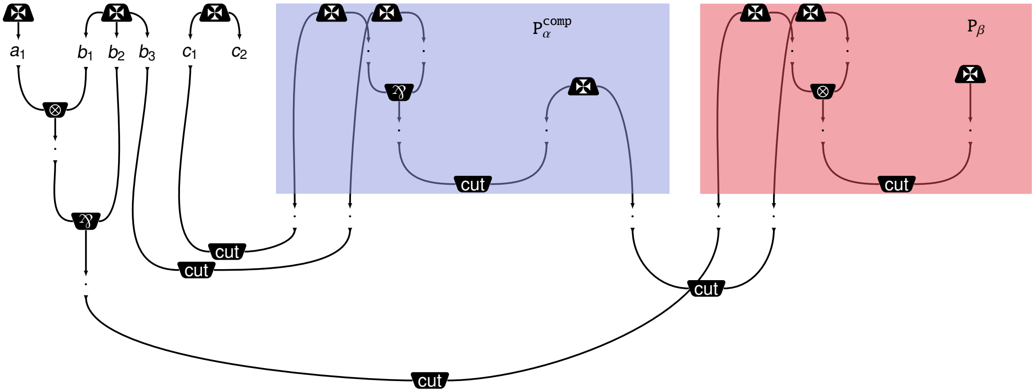
Creating Parsing Tests (1/6)



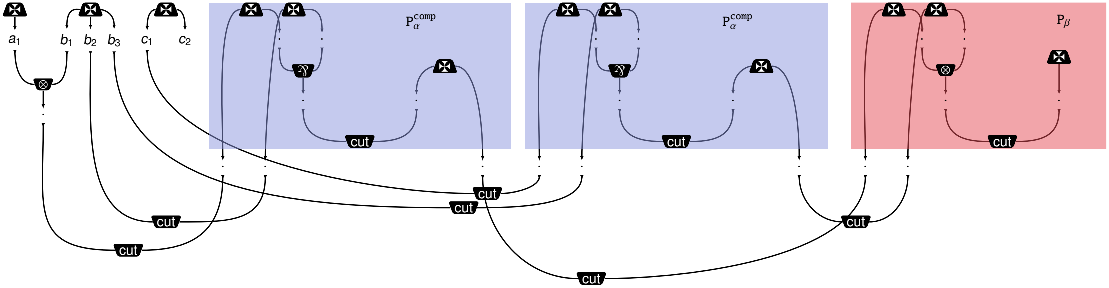
Creating Parsing Tests (2/6)



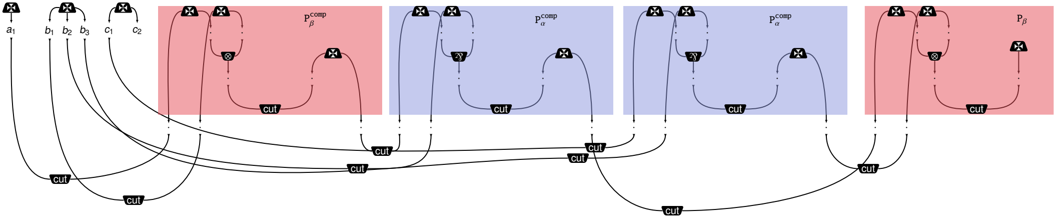
Creating Parsing Tests (3/6)



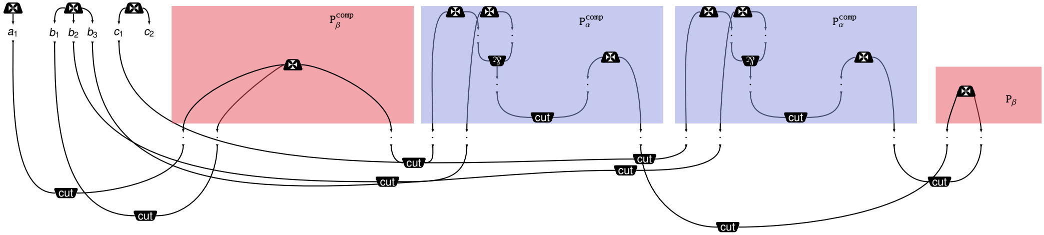
Creating Parsing Tests (4/6)



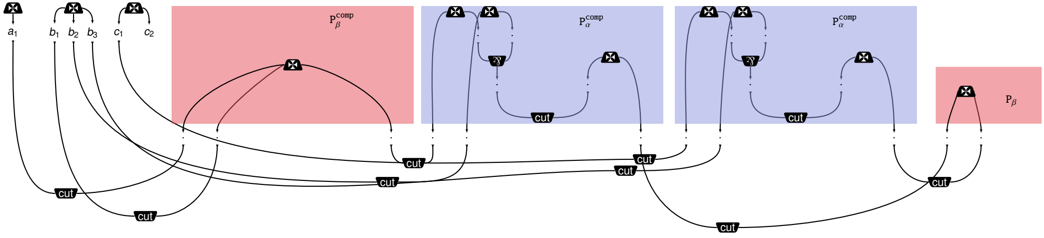
Creating Parsing Tests (5/6)



Creating Parsing Tests (6/6)

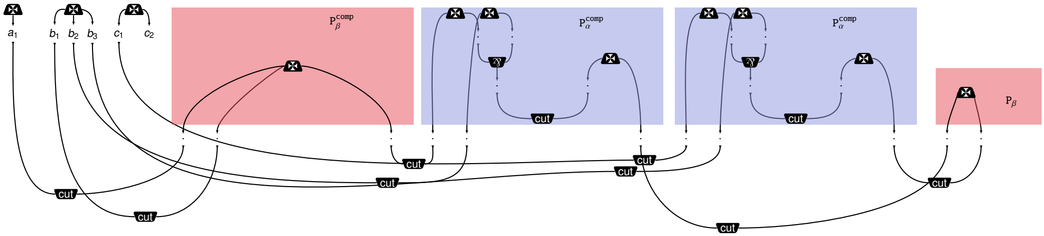


Why is it Linear? (1/5)



How it works:

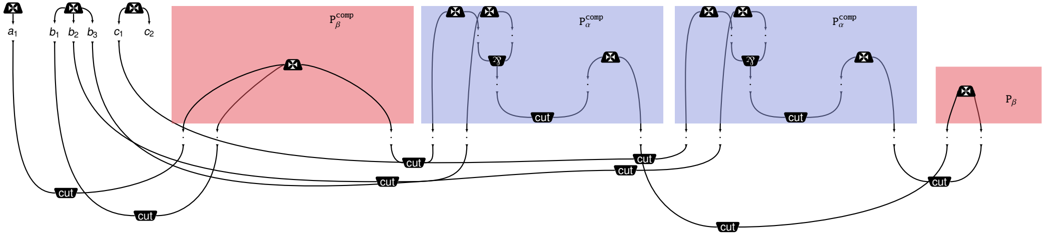
Why is it Linear? (2/5)



How it works:

1. Disconnections are **irreversible** w.r.t. cut–elimination.

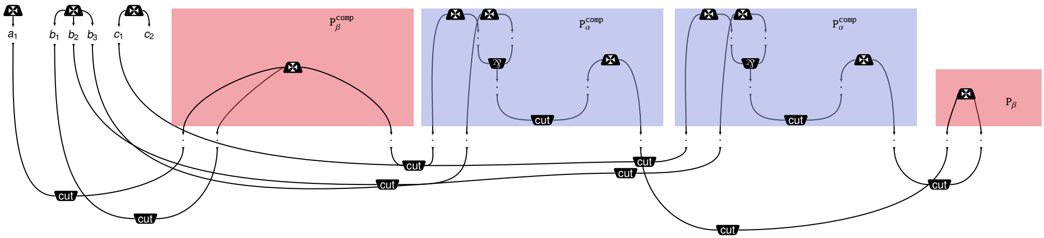
Why is it Linear? (3/5)



How it works:

1. Disconnections are **irreversible** w.r.t. cut–elimination.
2. Againts $\mathrm{P}_\alpha, \mathrm{P}_\beta$ cycles are **irreversible** w.r.t. cut–elimination.

Why is it Linear? (4/5)



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1. Disconnections are **irreversible** w.r.t. cut–elimination.
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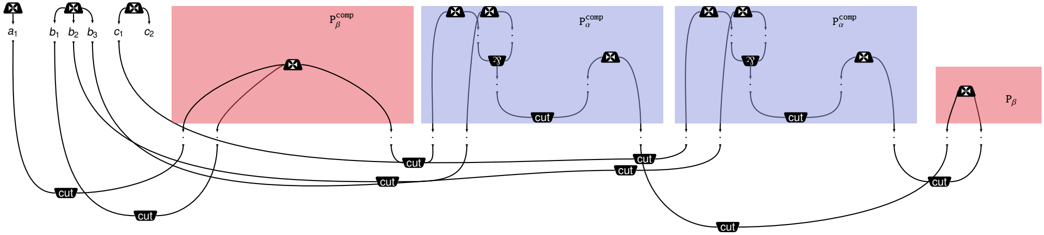
3. First eliminate all reversible cut.

Why is it Linear? (5/5)



How it works:

1. Disconnections are **irreversible** w.r.t. cut–elimination.
2. Againts $\mathrm{P}_\alpha, \mathrm{P}_\beta$ cycles are **irreversible** w.r.t. cut–elimination.
3. First eliminate all reversible cut.
4. Only ($\parr$/$\maltese$) cut remain, they must not create disconnections and so the choice **does not matter**.

Thank You