Untyped Net and Second Order Quantifiers

PACM$\wedge$N Workshop

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I – An overview of Realisability

The Brouwer–Heyting–Kolmogorov Interpretation

BHK Interpretation (1/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$$

BHK Interpretation (2/ 13)

$$\pi : A \wedge B \qquad \Leftrightarrow \qquad \pi = \langle \pi_1, \pi_2 \rangle$$

$\pi_1 : A$

BHK Interpretation (3/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \overset{\pi_1 \,:\, A}{\pi_1}, \overset{\pi_2 \,:\, B}{\pi_2} \rangle$$

BHK Interpretation (4/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \underset{\pi_1 : A}{\pi_1}, \underset{\pi_2 : B}{\pi_2} \rangle$$

$$\pi = \langle 0, \rho \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow$$

BHK Interpretation (5/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$$

$\pi_1 : A$ $\pi_2 : B$

$$\pi = \langle 0, \rho \rangle$$

$\rho : A$

$$\pi : A \vee B \quad \Leftrightarrow$$

BHK Interpretation (6/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \quad \text{where } \pi_1 : A,\ \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$$

OR

$$\pi = \langle 1, \rho \rangle \quad \rho : B$$

BHK Interpretation (7/ 13)

$$\pi : A \land B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \quad (\pi_1 : A,\ \pi_2 : B)$$

$$\pi : A \lor B \quad \Leftrightarrow \quad \begin{array}{l} \pi = \langle 0, \rho \rangle \quad (\rho : A) \\ \text{OR} \\ \pi = \langle 1, \rho \rangle \quad (\rho : B) \end{array}$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

BHK Interpretation (8/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : \exists x \in X\, Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$$

OR

$$\pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

BHK Interpretation (9/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \underset{\pi_1 : A}{\pi_1}, \underset{\pi_2 : B}{\pi_2} \rangle$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \underset{\rho : Px}{\rho} \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \underset{\rho : A}{\rho} \rangle \quad \text{OR} \quad \pi = \langle 1, \underset{\rho : B}{\rho} \rangle$$

$$\pi : A \rightarrow B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

BHK Interpretation (10/ 13)

$\pi_1 : A$ $\quad \pi_2 : B$

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$

$\rho : A$

$\pi = \langle 0, \rho \rangle$

$\pi : A \vee B \quad \Leftrightarrow \quad$ OR

$\rho : B$

$\pi = \langle 1, \rho \rangle$

$\pi : A \rightarrow B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\rho : Px$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

BHK Interpretation (11/ 13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$$

OR

$$\pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \quad \rho : Px$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot$$

BHK Interpretation (12/ 13)

$\pi_1 : A$ $\quad \pi_2 : B$

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$$

$\rho : A$

$$\pi = \langle 0, \rho \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \text{OR}$$

$\rho : B$

$$\pi = \langle 1, \rho \rangle$$

$$\pi : A \rightarrow B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$\rho : Px$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

BHK Interpretation (13/13)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A \qquad \text{OR} \qquad \pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \qquad \rho : Px$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

I – An overview of Realisability

Algebraic aspects of the BHK interpretation

Algebraic aspects in BHK (1/ 3)

$\pi_1 : A$ $\quad \pi_2 : B$

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$$

$\rho : A$

$$\pi = \langle 0, \rho \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \text{OR}$$

$\rho : B$

$$\pi = \langle 1, \rho \rangle$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$\rho : Px$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \perp \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \perp$$

Algebraic aspects in BHK (2/ 3)

$\pi_1 : A \quad \pi_2 : B$

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$

$\rho : A$

$\pi = \langle 0, \rho \rangle$

$\pi : A \vee B \quad \Leftrightarrow \quad$ OR

$\rho : B$

$\pi = \langle 1, \rho \rangle$

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\rho : Px$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

A notion of Interaction

Algebraic aspects in BHK (3/ 3)

$\pi_1 : A$ $\pi_2 : B$

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$

$\rho : A$

$\pi = \langle 0, \rho \rangle$

$\pi : A \vee B \quad \Leftrightarrow \quad$ OR

$\rho : B$

$\pi = \langle 1, \rho \rangle$

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\rho : Px$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

A notion of product

A notion of Interaction

I – An overview of Realisability

Realisability:
Implementing the BHK interpretation

BHK Implementations (1/ 14)

Formulas

Programs

BHK Implementations (2/ 14)

Formulas

Programs$\langle \cdot, \cdot \rangle$

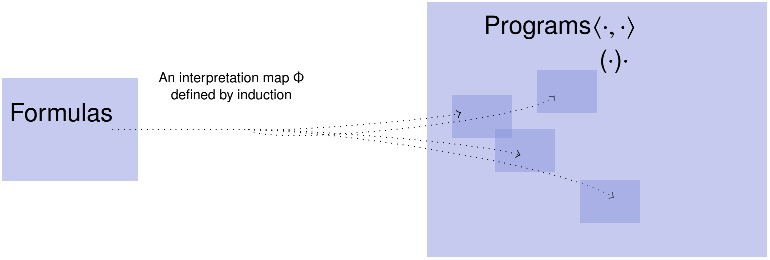
BHK Implementations (3/ 14)

Formulas

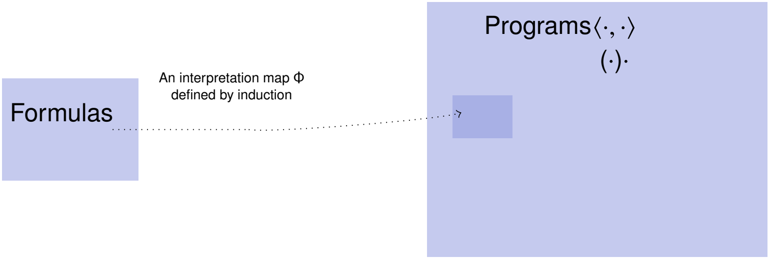
Programs$\langle \cdot, \cdot \rangle$

$(\cdot)_{\cdot}$

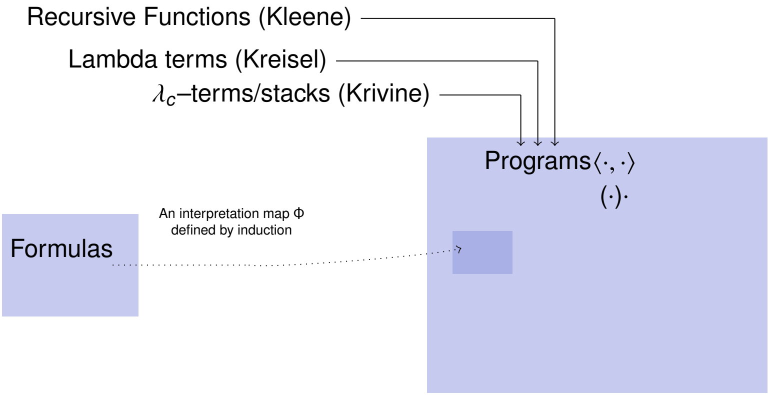
BHK Implementations (4/ 14)



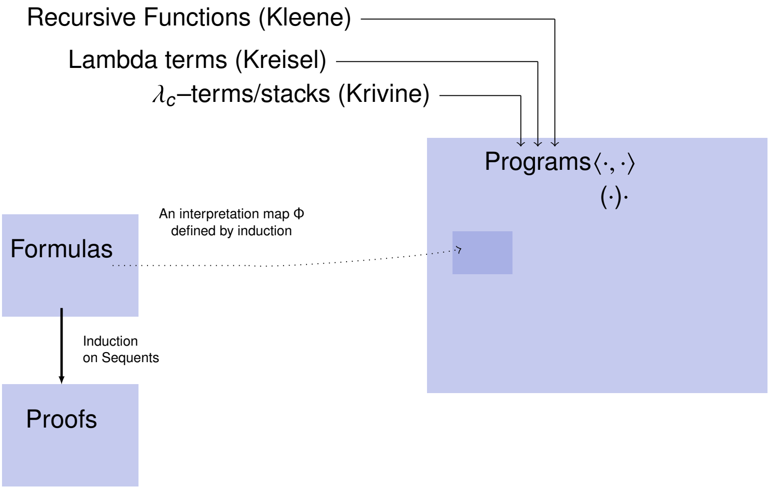
BHK Implementations (5/ 14)



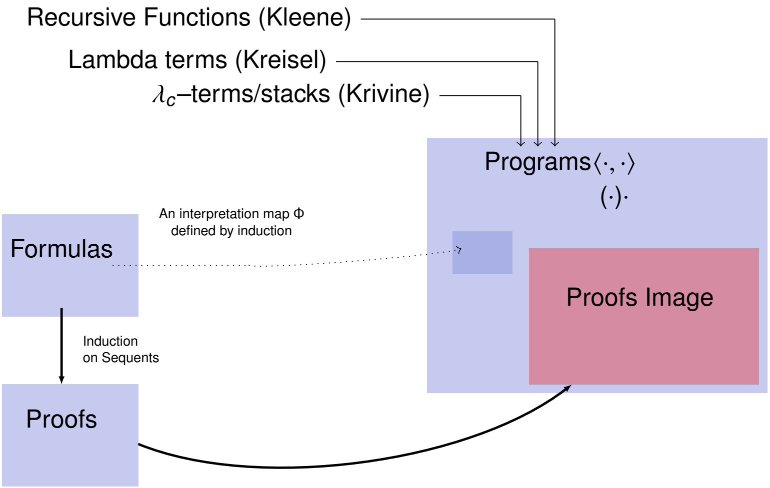
BHK Implementations (6/ 14)



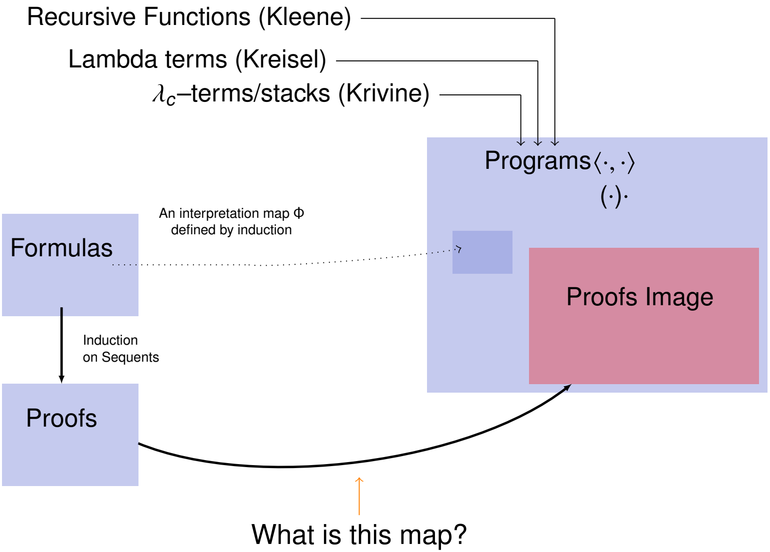
BHK Implementations (7/ 14)



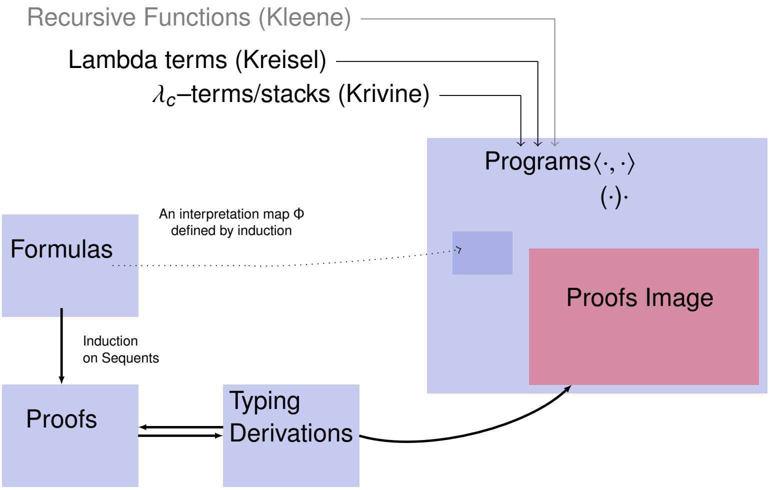
BHK Implementations (8/ 14)



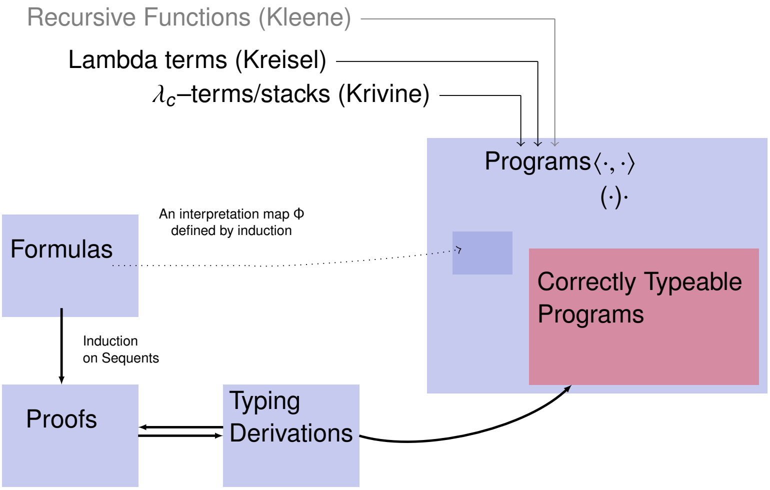
BHK Implementations (9/ 14)



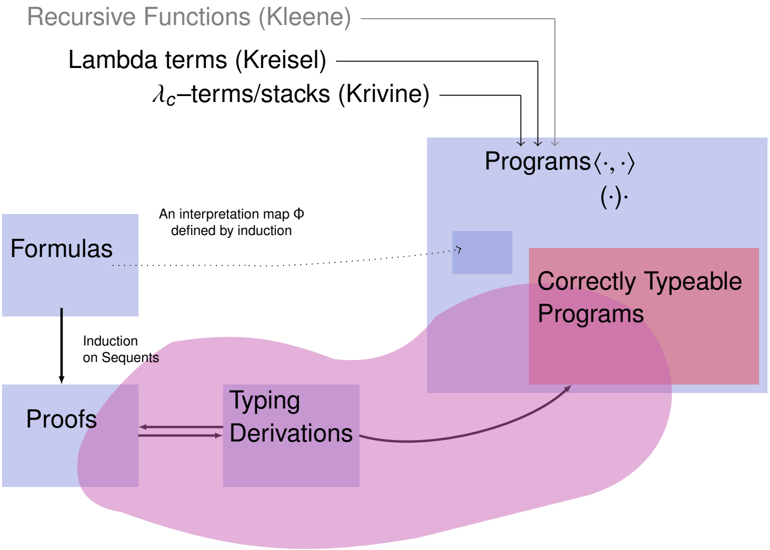
BHK Implementations (10/ 14)



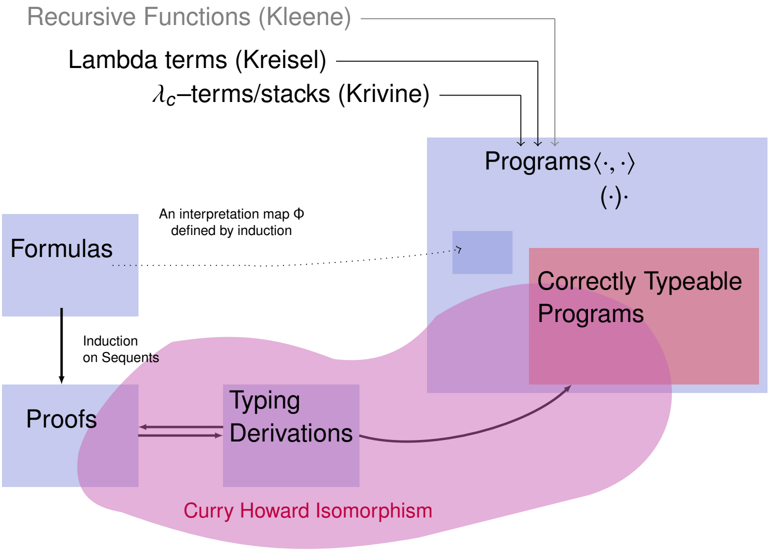
BHK Implementations (11/ 14)



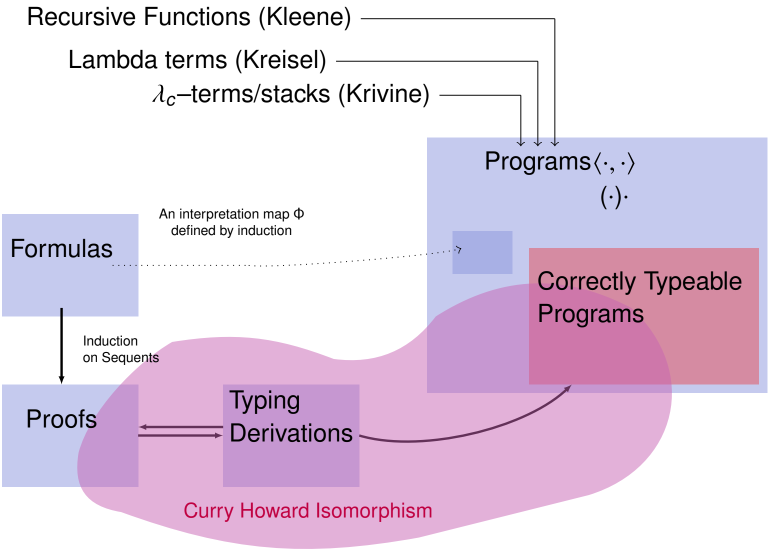
BHK Implementations (12/ 14)



BHK Implementations (13/ 14)



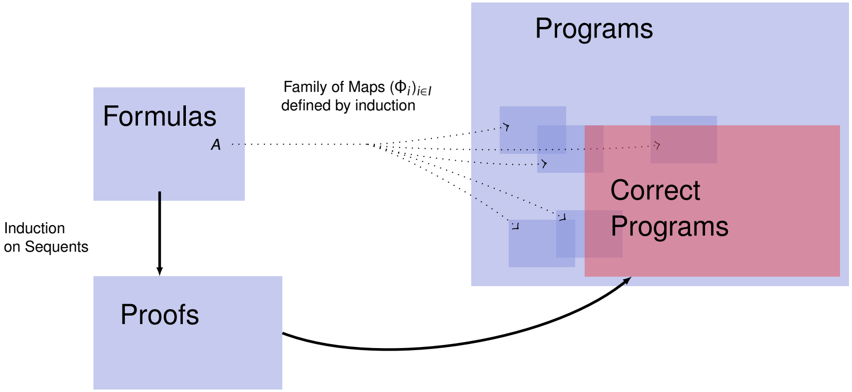
BHK Implementations (14/ 14)



I – An overview of Realisability

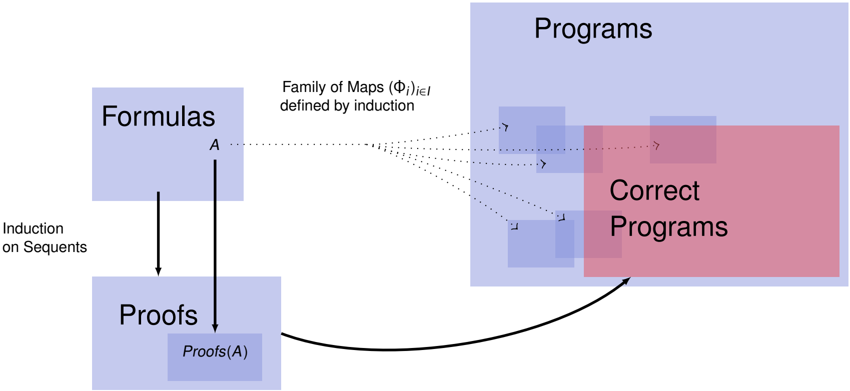
The Adequacy theorem

Adequacy (1/ 4)



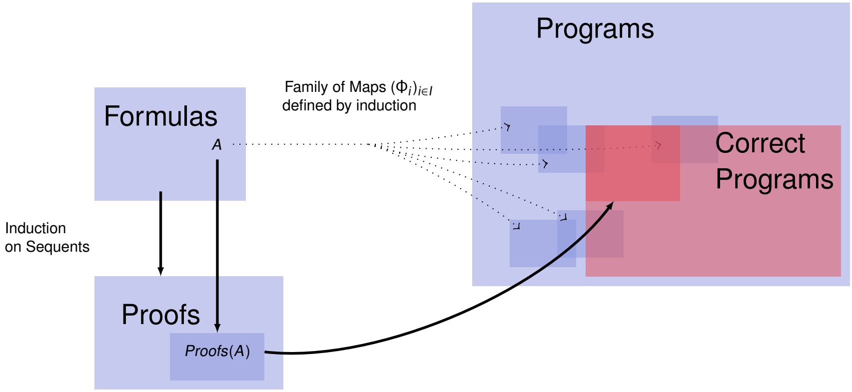
Programs
Formulas
A
Family of Maps $(\Phi_i)_{i \in I}$
defined by induction
Correct
Programs
Induction
on Sequents
Proofs

Adequacy (2/ 4)



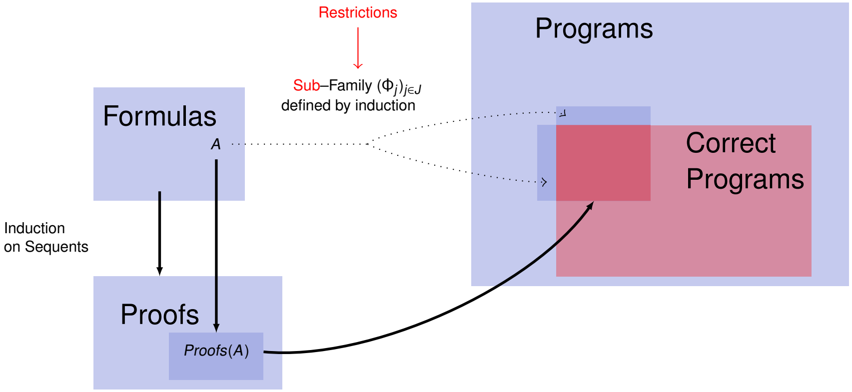
Programs
Formulas
A
Family of Maps $(\Phi_i)_{i \in I}$
defined by induction
Correct
Programs
Induction
on Sequents
Proofs
$Proofs(A)$

Adequacy (3/ 4)



Formulas
A
Family of Maps $(\Phi_i)_{i \in I}$
defined by induction
Programs
Correct
Programs
Induction
on Sequents
Proofs
$Proofs(A)$

Adequacy (4/ 4)



Restrictions
Sub–Family $(\Phi_j)_{j \in J}$
defined by induction
Programs
Formulas
A
Correct
Programs
Induction
on Sequents
Proofs
$Proofs(A)$

II – Interactive Realisability

The Limits to Consistency

Consistency (1/ 6)

Consistency

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False $\perp$ cannot be proved!

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Contradiction must be avoided!

$\Rightarrow$ No proof of $A \wedge \neg A$!

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$$\text{Proof}(A) \neq \emptyset \quad \Rightarrow \quad \text{Proof}(\neg A) = \emptyset$$

Consistency

False $\perp$ cannot be proved!

Contradiction must be avoided!
$\Rightarrow$ No proof of $A \wedge \neg A$!

$$\text{Proof}(A) \neq \emptyset \quad \Rightarrow \quad \text{Proof}(\neg A) = \emptyset$$

Consistency
inihibits
interaction

Consistency (6/ 6)

Consistency

False $\perp$ cannot be proved!

Contradiction must be avoided!
$\Rightarrow$ No proof of $A \wedge \neg A$!

$$\text{Proof}(A) \neq \emptyset \quad \Rightarrow \quad \text{Proof}(\neg A) = \emptyset$$

Consistency inihibits interaction

$$\dfrac{\begin{array}{c}\vdots\,\pi \\ A \Rightarrow \perp\end{array} \qquad \begin{array}{c}\vdots\,\rho \\ A\end{array}}{\perp}\ \text{cut}$$

Cannot exists!

II – Interactive Realisability

Consistency in the BHK interpretation

Consistency in BHK (1/4)

$$\pi : A \wedge B \iff \pi = \langle \overset{\pi_1 : A}{\pi_1}, \overset{\pi_2 : B}{\pi_2} \rangle$$

$$\pi : A \vee B \iff \pi = \langle 0, \overset{\rho : A}{\rho} \rangle \quad \text{OR} \quad \pi = \langle 1, \overset{\rho : B}{\rho} \rangle$$

$$\pi : A \to B \iff \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \iff \pi = \langle x, \overset{\rho : Px}{\rho} \rangle$$

$$\pi : \forall x \in X\ Px \iff \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \iff \text{None}$$

$$\pi : \neg A \iff \pi : A \to \bot$$

Consistency in BHK (2/ 4)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \underbrace{\pi_1}_{\pi_1 : A}, \underbrace{\pi_2}_{\pi_2 : B} \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \underbrace{\rho}_{\rho : A} \rangle \quad \text{OR} \quad \pi = \langle 1, \underbrace{\rho}_{\rho : B} \rangle$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X \; Px \quad \Leftrightarrow \quad \pi = \langle x, \underbrace{\rho}_{\rho : Px} \rangle$$

$$\pi : \forall x \in X \; Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

Consistency in BHK (3/ 4)

$\pi_1 : A$ $\quad \pi_2 : B$

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$$

$\rho : A$

$$\pi = \langle 0, \rho \rangle$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \text{OR}$$

$\rho : B$

$$\pi = \langle 1, \rho \rangle$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

Interaction

$\rho : Px$

$$\pi : \exists x \in X \; Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$$

$$\pi : \forall x \in X \; Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

Consistency in BHK (4/ 4)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$$

OR

$$\pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \qquad \rho : Px$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

Interaction

Cut Rule

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \Rightarrow B\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ A\end{matrix}}{B}\ \text{cut}$$

A limit to interaction (1/ 7)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$$

OR

$$\pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \qquad \rho : Px$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

Interaction

Cut Rule

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ A \to \bot\end{array} \qquad \begin{array}{c}\rho \\ \vdots \\ A\end{array}}{\bot}\ \text{cut}$$

A limit to interaction (2/ 7)

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \quad \pi_1 : A \quad \pi_2 : B$

$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$

OR

$\pi = \langle 1, \rho \rangle \quad \rho : B$

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \quad \rho : Px$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

Interaction

Cut Rule

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \to \bot\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ A\end{matrix}}{\bot}\ \text{cut}$$

A limit to interaction (3/ 7)

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$

$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A$ OR $\pi = \langle 1, \rho \rangle \quad \rho : B$

$\pi : A \rightarrow B \quad \Leftrightarrow$ for any $\rho : A \quad (\pi)\rho : B$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \qquad \rho : Px$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow$ for any $x \in X \quad (\pi)x : Px$

$\pi : \bot \quad \Leftrightarrow$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \rightarrow \bot$

No such proof can exists

Interaction

Cut Rule

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \rightarrow \bot\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ A\end{matrix}}{\bot}\ \text{cut}$$

A limit to interaction (4/ 7)

$\pi_1 : A$ $\quad \pi_2 : B$

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$

$\rho : Px$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$

$\rho : A$

$\pi = \langle 0, \rho \rangle$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : A \vee B \quad \Leftrightarrow \quad$ OR

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\rho : B$

$\pi = \langle 1, \rho \rangle$

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

Interaction

Cut Rule

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \to \bot\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ A\end{matrix}}{\bot}\ \text{cut}$$

No such ~~proof~~ realizer can exists

A limit to interaction (5/ 7)

$$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle \qquad \pi_1 : A \quad \pi_2 : B$$

$$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle \quad \rho : A \quad \text{OR} \quad \pi = \langle 1, \rho \rangle \quad \rho : B$$

$$\pi : A \rightarrow B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle \qquad \rho : Px$$

$$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad \text{for any } x \in X \quad (\pi)x : Px$$

$$\pi : \bot \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \rightarrow \bot$$

Interaction

Cut Rule

$$\dfrac{\overset{\pi}{\vdots} \atop A \rightarrow \bot \qquad \overset{\rho}{\vdots} \atop A}{\bot}\ \text{cut}$$

No such ~~proof~~ realizer can exists

$$[\![A]\!] \neq \emptyset \quad \Rightarrow \quad [\![A \rightarrow \bot]\!] = \emptyset$$

A limit to interaction (6/ 7)

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$ with $\pi_1 : A$, $\pi_2 : B$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$ with $\rho : Px$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : A \vee B \quad \Leftrightarrow \quad \pi = \langle 0, \rho \rangle$ with $\rho : A$ OR $\pi = \langle 1, \rho \rangle$ with $\rho : B$

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

Empty!

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

Interaction

$$\dfrac{\overset{\pi}{\vdots} \qquad\qquad \overset{\rho}{\vdots}}{\dfrac{A \to \bot \qquad\qquad A}{\bot}}\ \text{cut}$$

Cut Rule

No such ~~proof~~ realizer can exists

$[\![A]\!] \neq \emptyset \quad \Rightarrow \quad [\![A \to \bot]\!] = \emptyset$

A limit to interaction (7/ 7)

$\pi_1 : A$ $\quad$ $\pi_2 : B$

$\pi : A \wedge B \quad \Leftrightarrow \quad \pi = \langle \pi_1, \pi_2 \rangle$

$\rho : A$

$\pi = \langle 0, \rho \rangle$

$\pi : A \vee B \quad \Leftrightarrow \quad$ OR

$\rho : B$

$\pi = \langle 1, \rho \rangle$

Empty!

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\rho : Px$

$\pi : \exists x \in X\ Px \quad \Leftrightarrow \quad \pi = \langle x, \rho \rangle$

$\pi : \forall x \in X\ Px \quad \Leftrightarrow \quad$ for any $x \in X \quad (\pi)x : Px$

$\pi : \bot \quad \Leftrightarrow \quad$ None

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

No such ~~proof~~ realizer can exists

$[\![A]\!] \neq \emptyset \quad \Rightarrow \quad [\![A \to \bot]\!] = \emptyset$

$[\![A]\!] = \emptyset \quad \Rightarrow \quad [\![A \to \bot]\!] = \text{ALL}$

Interaction

$$\dfrac{\overset{\pi}{\vdots} \atop A \to \bot \qquad \overset{\rho}{\vdots} \atop A}{\bot}\ \text{cut}$$

Cut Rule

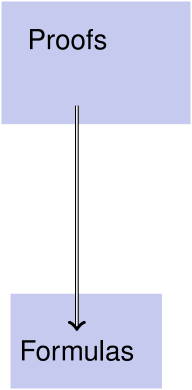
II – Interactive Realisability

Proofs and Counter Proofs : Breaking Consistency

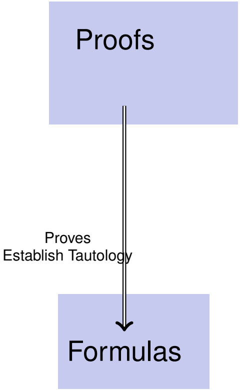
Proofs and Models (1/ 5)

Formulas

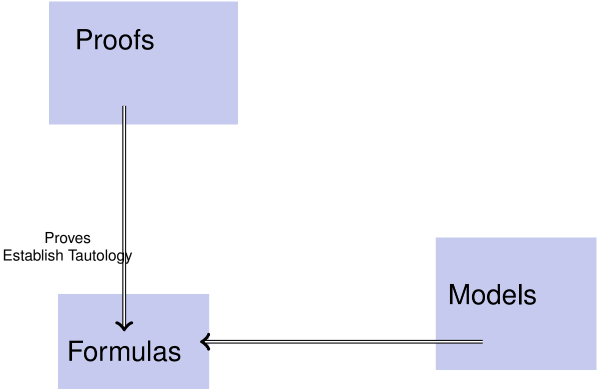
Proofs and Models (2/ 5)



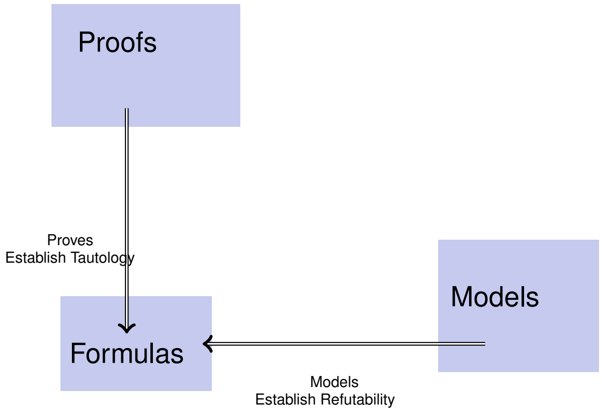
Proofs and Models (3/ 5)

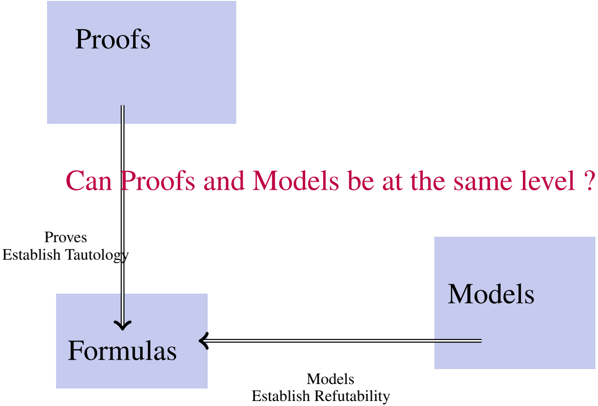


Proofs and Models (4/ 5)



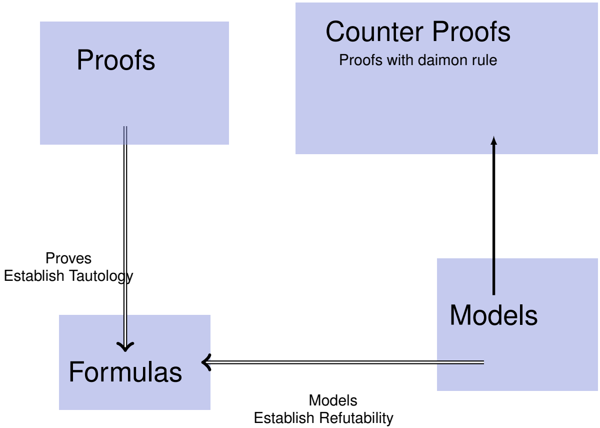
Proofs and Models (5/ 5)



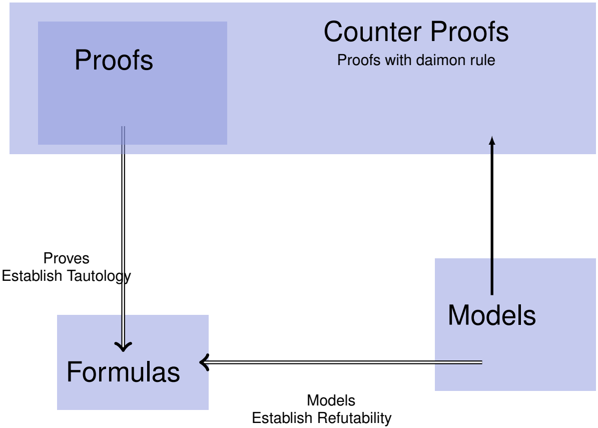


Can Proofs and Models be at the same level ?

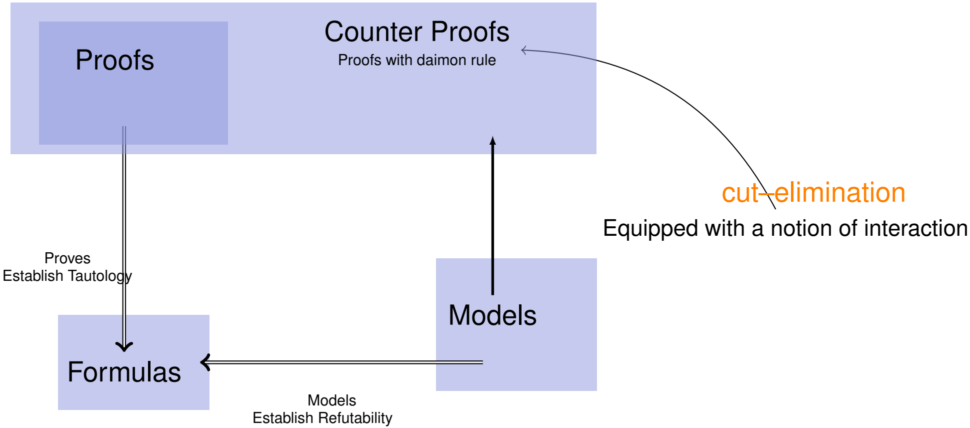
Counter proofs (1/ 3)



Counter proofs (2/ 3)



Counter proofs (3/ 3)



II – Interactive Realisability

Proofs and Counter Proofs : Breaking Consistency

Towards an interactive framework (1/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \perp \quad \Leftrightarrow \quad \text{None}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \perp$$

$$\dfrac{\begin{matrix}\pi \,\vdots \\ A \Rightarrow \perp\end{matrix} \qquad\qquad \begin{matrix}\rho \,\vdots \\ A\end{matrix}}{\perp}\ \text{cut}$$

cannot exists

Towards an interactive framework (2/ 10)

$$\pi : A \rightarrow B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \perp \quad \Leftrightarrow \quad \xcancel{\text{None}}$$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \rightarrow \perp$$

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ A \Rightarrow \perp\end{array} \qquad \begin{array}{c}\rho \\ \vdots \\ A\end{array}}{\perp}\;\text{cut}$$

Towards an interactive framework (3/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \bot \quad \Leftrightarrow \quad \cancel{\text{None}}$$

The pole $[\![\bot]\!] \neq \emptyset$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

$$\dfrac{\begin{array}{c}\pi \vdots \\ A \Rightarrow \bot\end{array} \qquad \begin{array}{c}\rho \vdots \\ A\end{array}}{\bot}\ \text{cut}$$

Towards an interactive framework (4/ 10)

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\pi : \bot \quad \Leftrightarrow \quad$ ~~None~~

The pole $[\![\bot]\!] \neq \emptyset$

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

$$\dfrac{\begin{matrix}\pi \vdots \\ A \Rightarrow \bot\end{matrix} \qquad \begin{matrix}\rho \vdots \\ A\end{matrix}}{\bot}\ \text{cut} \quad \overrightarrow{\text{becomes}} \quad \left(\begin{matrix}\pi \vdots \\ A \Rightarrow \bot\end{matrix}\right) \begin{matrix}\rho \vdots \\ A\end{matrix} \quad \in [\![\bot]\!]$$

Towards an interactive framework (5/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \bot \quad \Leftrightarrow \quad \cancel{\text{None}}$$

The pole $[\![\bot]\!] \neq \emptyset$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ A \Rightarrow \bot\end{array} \qquad \begin{array}{c}\rho \\ \vdots \\ ??\end{array}}{\bot}\ \text{cut} \quad \underset{\longrightarrow}{\text{becomes}} \quad \left(\begin{array}{c}\pi \\ \vdots \\ A \Rightarrow \bot\end{array}\right) \begin{array}{c}\rho \\ \vdots \\ ??\end{array} \in [\![\bot]\!]$$

Towards an interactive framework (6/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \bot \quad \Leftrightarrow \quad \cancel{\text{None}}$$

The pole $[\![\bot]\!] \neq \emptyset$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \bot\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ ??\end{matrix}}{\bot}\,\text{cut} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \bot\end{matrix}\right) \begin{matrix}\rho \\ \vdots \\ ??\end{matrix} \in [\![\bot]\!] \qquad \Longrightarrow \quad \rho : A$$

Towards an interactive framework (7/ 10)

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\pi : \bot \quad \Leftrightarrow \quad$ ~~None~~

The pole $[\![\bot]\!] \neq \emptyset$

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$

$$\dfrac{\begin{array}{c}\pi \vdots \\ A \Rightarrow \bot\end{array} \qquad \begin{array}{c}\rho \vdots \\ ??\end{array}}{\bot}\ \text{cut} \quad \xrightarrow{\text{becomes}} \quad \boxed{\left(\begin{array}{c}\pi \vdots \\ A \Rightarrow \bot\end{array}\right) \begin{array}{c}\rho \vdots \\ ??\end{array} \in [\![\bot]\!]} \quad \Longrightarrow \quad \rho : A$$

Towards an interactive framework (8/ 10)

$\pi : A \to B \quad \Leftrightarrow \quad$ for any $\rho : A \quad (\pi)\rho : B$

$\pi : \perp \quad \Leftrightarrow \quad$ ~~None~~

The pole $[\![\perp]\!] \neq \emptyset$

$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \perp$

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \perp\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ ??\end{matrix}}{\perp}\ \text{cut} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \perp\end{matrix}\right)\begin{matrix}\rho \\ \vdots \\ ??\end{matrix} \in [\![\perp]\!] \quad \Longrightarrow \quad \rho : A$$

$$\downarrow$$

$$(t)u \in [\![\perp]\!]$$

Towards an interactive framework (9/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \bot \quad \Leftrightarrow \quad \cancel{\text{None}}$$

The pole $[\![\bot]\!] \neq \emptyset$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \bot\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ ??\end{matrix}}{\bot}\ \text{cut} \quad \xrightarrow{\text{becomes}} \quad \boxed{\left(\begin{matrix}\pi \\ \vdots \\ A \Rightarrow \bot\end{matrix}\right)\begin{matrix}\rho \\ \vdots \\ ??\end{matrix} \in [\![\bot]\!]} \quad \Rightarrow \quad \rho : A$$

$$\Big\downarrow$$

$$\boxed{(t)u \in [\![\bot]\!] \quad \Leftrightarrow_{def} \quad t \bot u}$$

Towards an interactive framework (10/ 10)

$$\pi : A \to B \quad \Leftrightarrow \quad \text{for any } \rho : A \quad (\pi)\rho : B$$

$$\pi : \bot \quad \Leftrightarrow \quad \cancel{\text{None}}$$

The pole $[\![\bot]\!] \neq \emptyset$

$$\pi : \neg A \quad \Leftrightarrow \quad \pi : A \to \bot$$

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ A \Rightarrow \bot\end{array} \qquad \begin{array}{c}\rho \\ \vdots \\ ??\end{array}}{\bot}\ \text{cut} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{c}\pi \\ \vdots \\ A \Rightarrow \bot\end{array}\right) \begin{array}{c}\rho \\ \vdots \\ ??\end{array} \in [\![\bot]\!] \quad \Longrightarrow \quad \rho : A$$

$$\downarrow$$

$$(t)u \in [\![\bot]\!] \quad \Leftrightarrow_{def} \quad t \perp u$$

Orthogonality

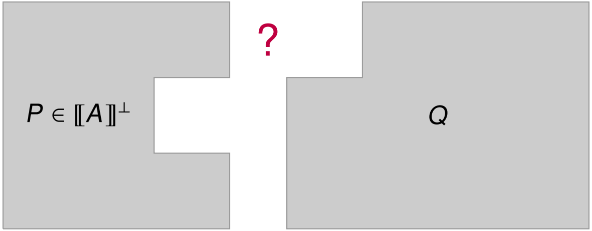
II – Interactive Realisability

Orthogonality in realisability models

Types in Orthogonality models (1/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

$([\![A]\!] = [\![A]\!]^{\perp\perp})$

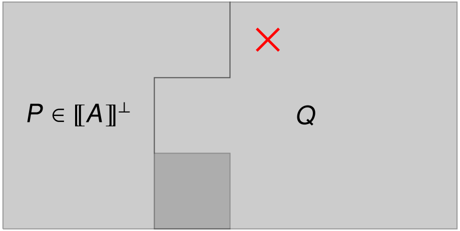


Does Q belong to $[\![A]\!]$?

Types in Orthogonality models (2/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

$([\![A]\!] = [\![A]\!]^{\perp\perp})$

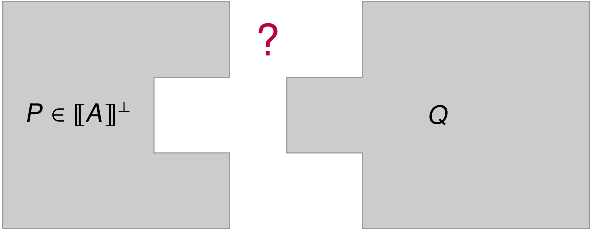


Q fails interaction $\Rightarrow$ $Q \notin [\![A]\!]$

Types in Orthogonality models (3/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$

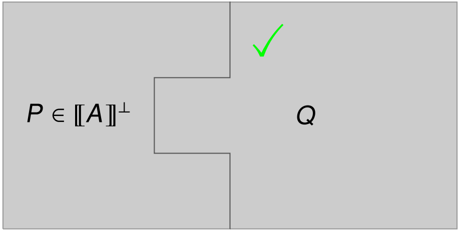
$([\![A]\!] = [\![A]\!]^{\perp\perp})$



Does Q belong to $[\![A]\!]$?

Types in Orthogonality models (4/4)

Realise A = Orthogonal to $[\![A]\!]^{\perp}$
$([\![A]\!] = [\![A]\!]^{\perp\perp})$

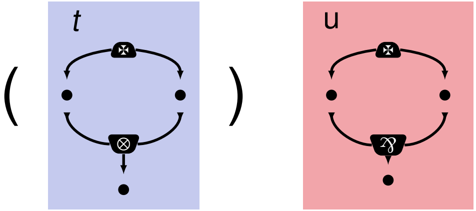


Infer $Q \in [\![A]\!]$.

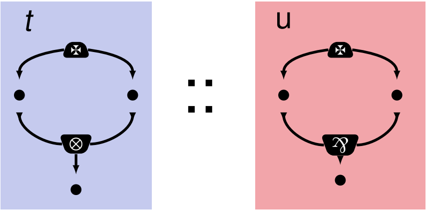
II – Interactive Realisability

Example with nets

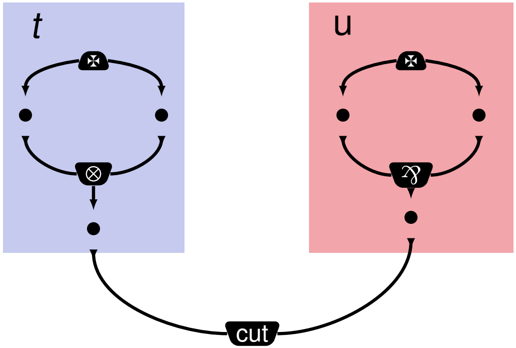
Cyclic interaction (1/7)



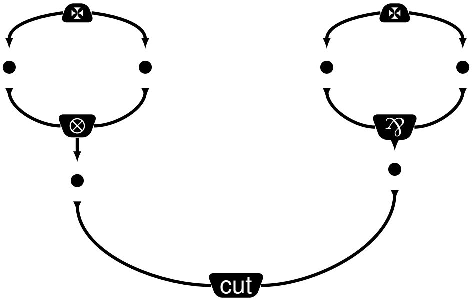
Cyclic interaction (2/7)



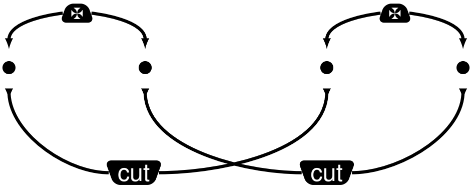
Cyclic interaction (3/ 7)



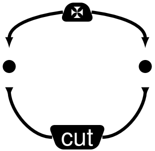
Cyclic interaction (4/ 7)



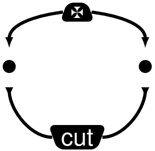
Cyclic interaction (5/ 7)



Cyclic interaction (6/ 7)



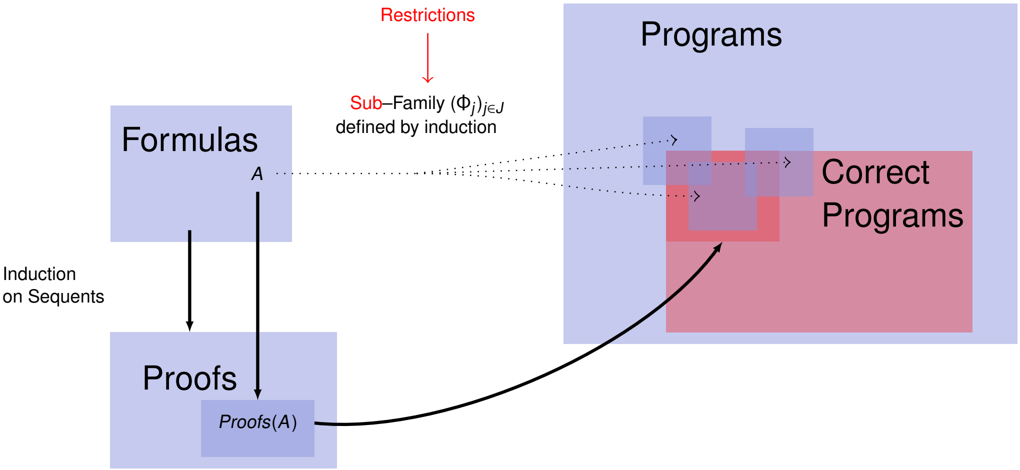
Cyclic interaction (7/ 7)



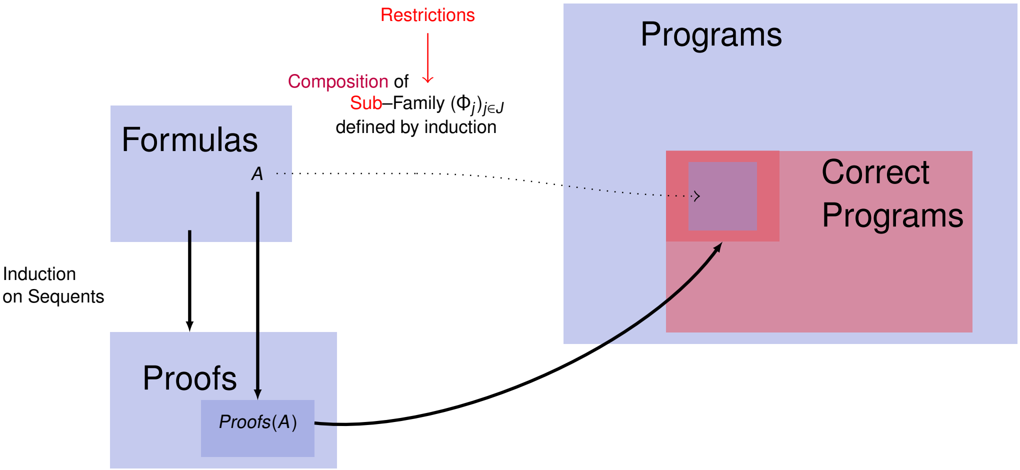
$\in [\![\perp]\!]$?

III – Completeness in Realizability

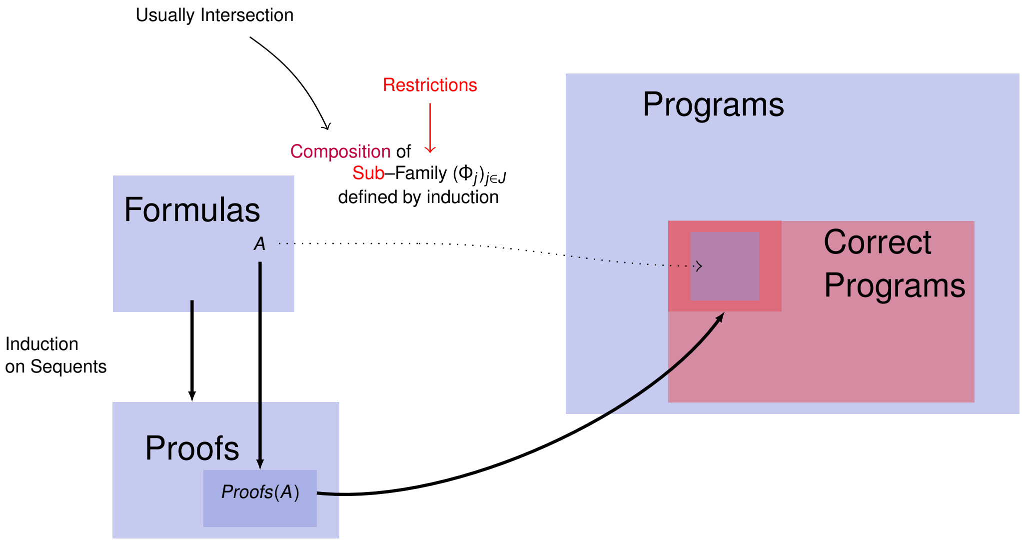
Completeness (1/ 4)



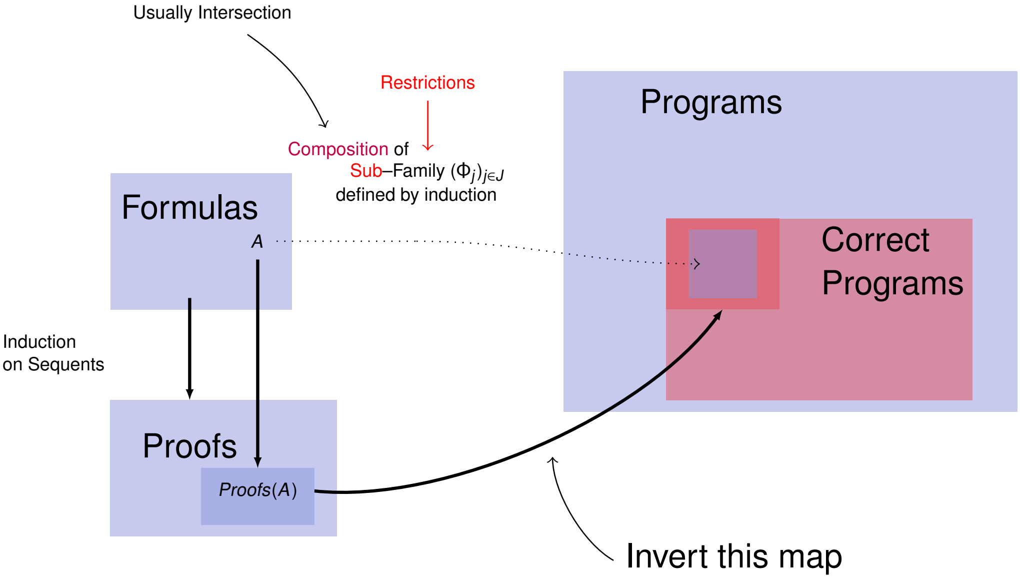
Completeness (2/ 4)



Completeness (3/ 4)



Completeness (4/ 4)



IV – Realisability for Linear Logic

The Proof System

Second Order Multiplicative Linear Logic

$$\frac{}{A, A^{\perp}}\ \mathrm{ax} \qquad \frac{}{\Gamma}\ \maltese$$

$$\frac{\Gamma, A \quad \Delta, A^{\perp}}{\Gamma, \Delta}\ \mathrm{cut} \qquad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta}\ \mathrm{ex}$$

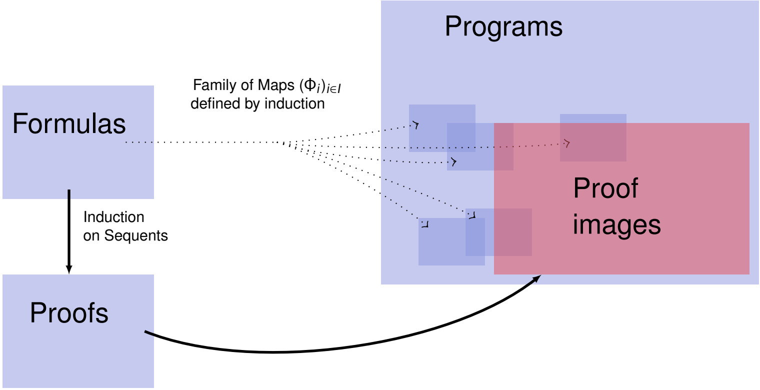
$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr$$

$$\frac{\Gamma, A[X \leftarrow B]}{\Gamma, \exists X A}\ \exists \qquad \frac{\Gamma, A}{\Gamma, \forall X A}\ \forall$$

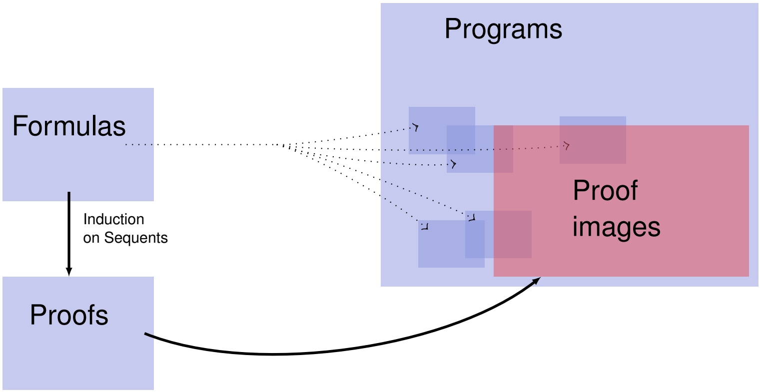
IV – Realisability for Linear Logic

Proof Structures: The Space of Realisers

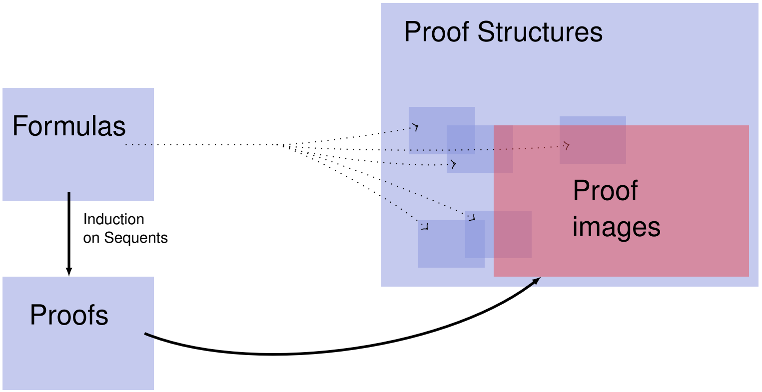
Linear Realisability (1/ 9)



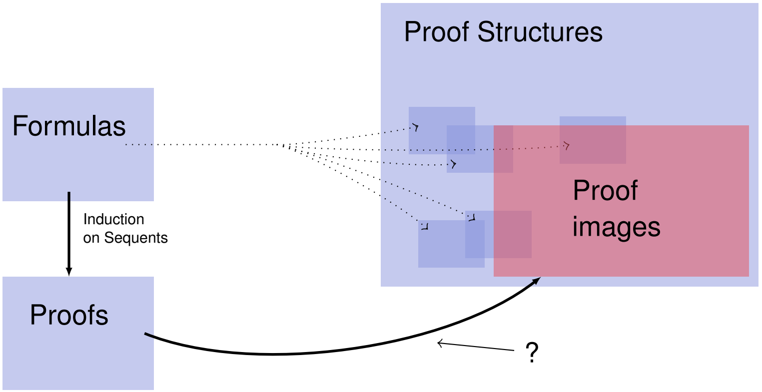
Linear Realisability (2/ 9)



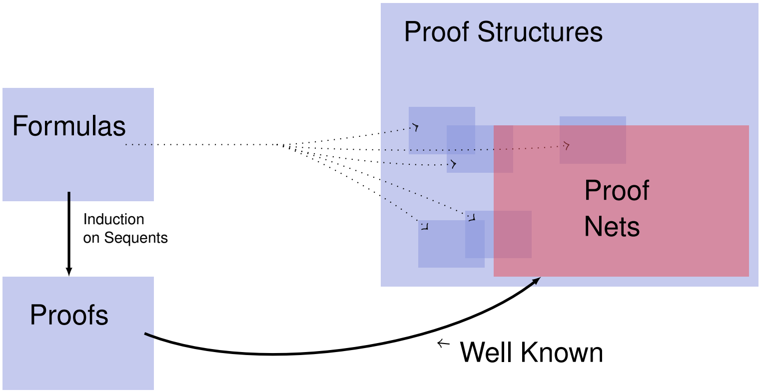
Linear Realisability (3/ 9)



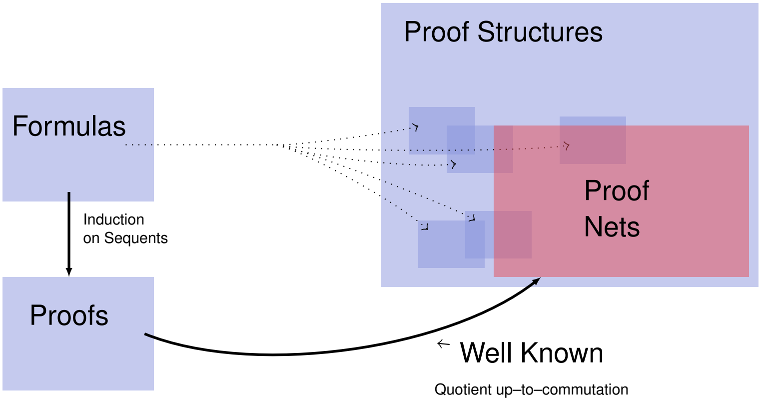
Linear Realisability (4/ 9)



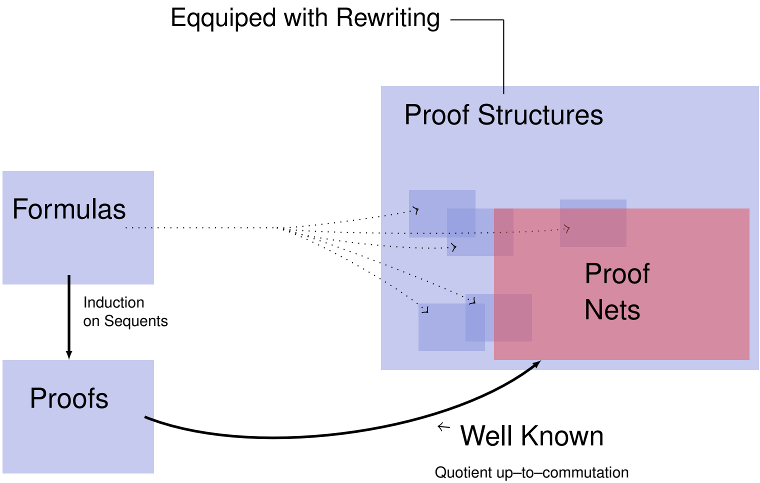
Linear Realisability (5/ 9)



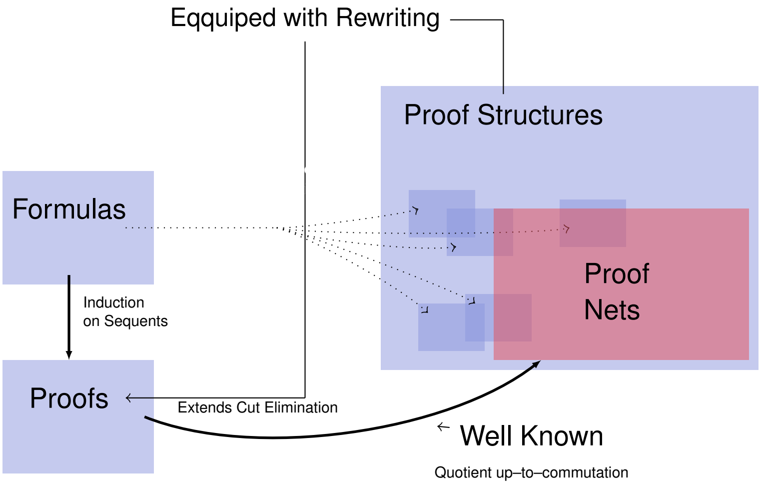
Linear Realisability (6/ 9)



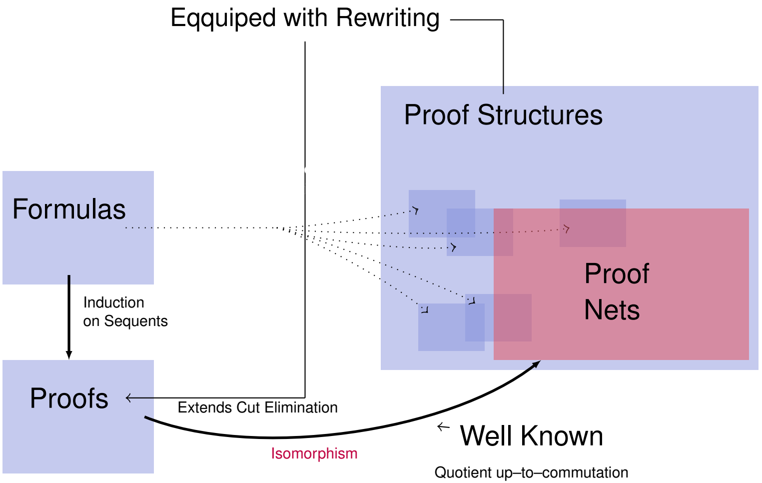
Linear Realisability (7/ 9)



Linear Realisability (8/ 9)



Linear Realisability (9/ 9)



IV – Realisability for Linear Logic

Current Results

Our results

We obtained adequacy and completeness results;

Theorem. (Adequacy). For any interpretation basis $\mathscr{B}$, any sequent Γ; $S \vdash \Gamma \Rightarrow S \in [\![\Gamma]\!]_{\mathscr{B}}$.

Theorem. (Completeness). $S \in \bigcap_{\mathscr{B}} [\![\Gamma]\!]_{\mathscr{B}} \Rightarrow S \vdash \Gamma$.

How to obtain a completeness result for MLL_2?

V – Nets for MLL_2

Directed Hypergraphs

$$(V, E, \mathsf{src}, \mathsf{tgt}, \ell)$$

Set of Vertices

Set of Edges

A source map $\mathsf{src} : E \to \mathcal{P}^{\leq}_{fin}(V)$
associates to each edge its sequence of sources

A target map $\mathsf{tgt} : E \to \mathcal{P}^{\leq}_{fin}(V)$
associates to each edge its sequence of targets

A labelling map $\ell : E \to L$

$$(V_1, E_1, \mathrm{src}_1, \mathrm{tgt}_1, \ell_1) + (V_2, E_2, \mathrm{src}_2, \mathrm{tgt}_2, \ell_2)$$

$$\triangleq$$

$$(V_1 \cup V_2, E_1 \uplus E_2, \mathrm{src}_1 \uplus \mathrm{src}_2, \mathrm{tgt}_1 \uplus \mathrm{tgt}_2, \ell_1 \uplus \ell_2)$$

Vertices may overlap!

Rename if necessary

Hyperedge/Link notation (1/ 6)

$$\langle a_1, \ldots, a_n \triangleright_c b_1, \ldots, b_k \rangle$$

Hyperedge/Link notation (2/ 6)

$$\langle a_1, \ldots, a_n \triangleright_c b_1, \ldots, b_k \rangle \qquad \triangleq \qquad (\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \{e\}, \mathrm{src}, \mathrm{tgt}, \ell)$$

Hyperedge/Link notation (3/ 6)

$$\langle \overbrace{a_1, \ldots, a_n}^{\mathrm{src}(e)} \triangleright_c b_1, \ldots, b_k \rangle \qquad \triangleq \qquad (\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \{e\}, \mathrm{src}, \mathrm{tgt}, \ell)$$

Hyperedge/Link notation (4/ 6)

$$\langle \overset{\mathrm{src}(e)}{\overset{\downarrow}{a_1, \ldots, a_n}} \triangleright_c \overset{\mathrm{tgt}(e)}{\overset{\downarrow}{b_1, \ldots, b_k}} \rangle \qquad \triangleq \qquad (\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \{e\}, \mathrm{src}, \mathrm{tgt}, \ell)$$

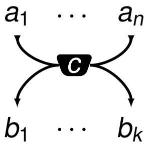
Hyperedge/Link notation (5/ 6)

$$\langle \overset{\mathrm{src}(e)}{\overset{\downarrow}{a_1, \ldots, a_n}} \triangleright_{\overset{\ell(e)}{\overset{\downarrow}{c}}} \overset{\mathrm{tgt}(e)}{\overset{\downarrow}{b_1, \ldots, b_k}} \rangle \qquad \triangleq \qquad (\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \{e\}, \mathrm{src}, \mathrm{tgt}, \ell)$$

Hyperedge/Link notation (6/ 6)

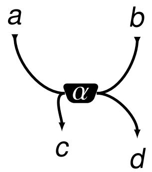
$$\langle \overbrace{a_1,\ldots,a_n}^{\mathrm{src}(e)} \triangleright_{\overset{\ell(e)}{c}} \overbrace{b_1,\ldots,b_k}^{\mathrm{tgt}(e)} \rangle \qquad \triangleq \qquad (\{a_1,\ldots,a_n,b_1,\ldots,b_n\},\{e\},\mathrm{src},\mathrm{tgt},\ell)$$

Represented as



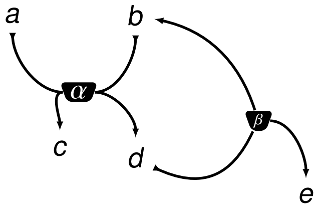
Describing hypergraphs (1/ 4)

$\langle a, b \;\triangleright_{\alpha}\; c, d \rangle$



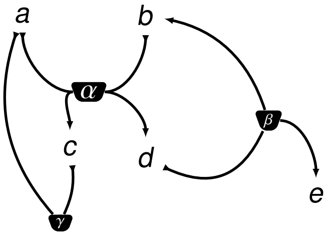
Describing hypergraphs (2/ 4)

$$\langle a, b \;\triangleright_\alpha\; c, d\rangle$$
$$+$$
$$\langle d \;\triangleright_\beta\; b, e\rangle$$



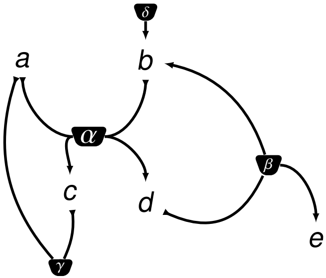
Describing hypergraphs (3/ 4)

$$\langle a, b \triangleright_\alpha c, d \rangle$$
$$+$$
$$\langle d \triangleright_\beta b, e \rangle$$
$$+$$
$$\langle a, c \triangleright_\gamma \rangle$$



Describing hypergraphs (4/ 4)

$$
\begin{gathered}
\langle a, b \triangleright_{\alpha} c, d \rangle \\
+ \\
\langle d \triangleright_{\beta} b, e \rangle \\
+ \\
\langle a, c \triangleright_{\gamma} \rangle \\
+ \\
\langle \triangleright_{\delta} b \rangle
\end{gathered}
$$



Properties of hypergraphs

Given $\mathcal{H} = (V, E, \mathrm{src}, \mathrm{tgt}, \ell)$ Labelset $= \{\maltese, \mathrm{cut}, \otimes, \parr, \forall, \exists\}$

$$\mathrm{tgt}(\mathcal{H}) \triangleq \bigcup_{e \in E} \mathrm{tgt}(e)$$

$$\mathrm{tgt}(\mathcal{H}) \triangleq \bigcup_{e \in E} \mathrm{tgt}(e)$$

Properties

source–disjoint for any $e \neq e' \in E$ $\mathrm{src}(e) \cap \mathrm{src}(e') = \emptyset$

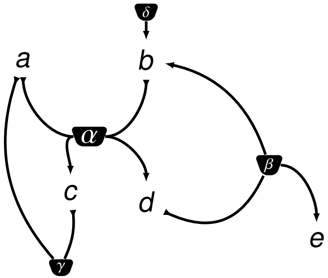
target–disjoint for any $e \neq e' \in E$ $\mathrm{tgt}(e) \cap \mathrm{tgt}(e') = \emptyset$

target–surjective $V = \mathrm{tgt}(\mathcal{H})$

$\mathcal{H}$ modular $\triangleq$ $\mathcal{H}$ source–disjoint, target–disjoint, target–surjective

Modular hypergraph (1/ 8)

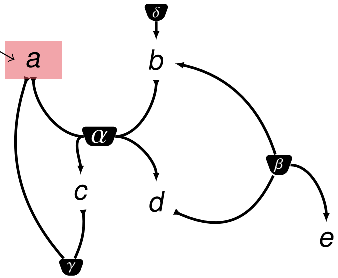
$$\begin{gathered}\langle a, b \;\triangleright_\alpha\; c, d\rangle \\ + \\ \langle d \;\triangleright_\beta\; b, e\rangle \\ + \\ \langle a, c \triangleright_\gamma \;\rangle \\ + \\ \langle \;\triangleright_\delta b\rangle\end{gathered}$$



Modular hypergraph (2/ 8)

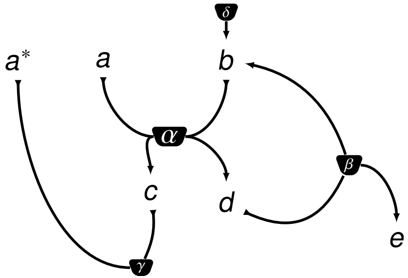
not source disjoint

$$\langle a, b \triangleright_{\alpha} c, d\rangle$$
$$+$$
$$\langle d \triangleright_{\beta} b, e\rangle$$
$$+$$
$$\langle a, c \triangleright_{\gamma} \rangle$$
$$+$$
$$\langle \triangleright_{\delta} b\rangle$$



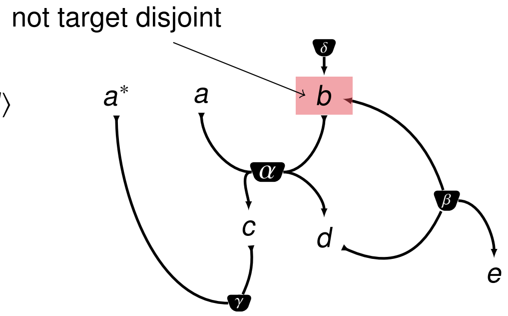
Modular hypergraph (3/ 8)

$$\langle a, b \ \triangleright_\alpha \ c, d\rangle$$
$$+$$
$$\langle d \ \triangleright_\beta \ b, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \ \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ b\rangle$$



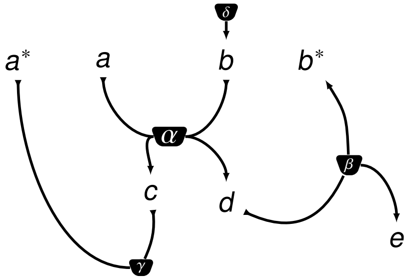
Modular hypergraph (4/ 8)

$$\langle a, b \ \triangleright_\alpha \ c, d\rangle$$
$$+$$
$$\langle d \ \triangleright_\beta \ b, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \ \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ b\rangle$$



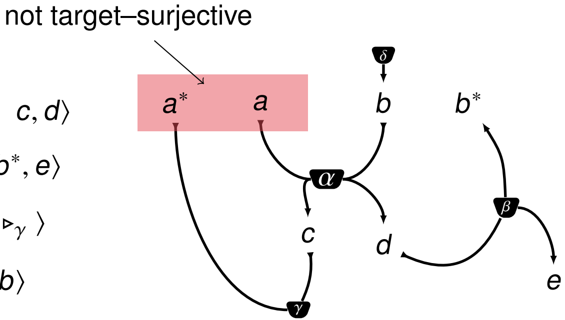
Modular hypergraph (5/ 8)

$$\langle a, b \triangleright_\alpha c, d\rangle$$
$$+$$
$$\langle d \triangleright_\beta b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \rangle$$
$$+$$
$$\langle \triangleright_\delta b\rangle$$



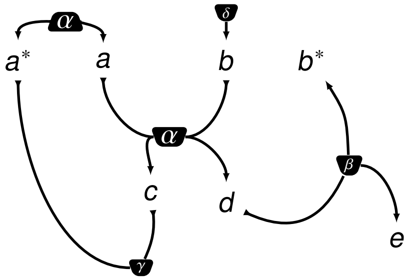
Modular hypergraph (6/ 8)

$$\langle a, b \;\triangleright_{\alpha}\; c, d \rangle$$
$$+$$
$$\langle d \;\triangleright_{\beta}\; b^{*}, e \rangle$$
$$+$$
$$\langle a^{*}, c \triangleright_{\gamma} \;\rangle$$
$$+$$
$$\langle \;\triangleright_{\delta}\; b \rangle$$



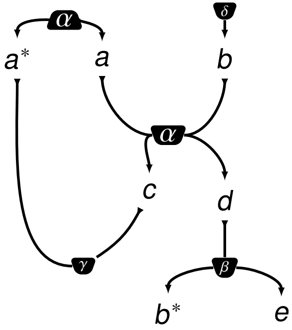
Modular hypergraph (7/ 8)

$$\langle a, b \ \triangleright_\alpha \ c, d\rangle$$
$$+$$
$$\langle d \ \triangleright_\beta \ b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \ \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ b\rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ a, a^*\rangle$$



Modular hypergraph (8/ 8)

$$\langle a, b \ \triangleright_\alpha \ c, d \rangle$$
$$+$$
$$\langle d \ \triangleright_\beta \ b^*, e \rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \ \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ b \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ a, a^* \rangle$$

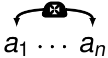


V – Nets for MLL_2

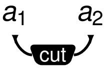
Generating the set of nets

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

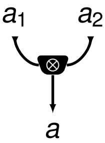
$\mathcal{H}$ net $\triangleq$ module + target–surjective



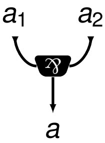
$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$



$\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$



$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$



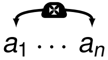
$\langle a_1, a_2 \triangleright_{\parr} a \rangle$

$\langle a \triangleright_{\forall} b \rangle$

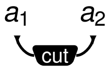
$\langle a \triangleright_{\exists} b \rangle$

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

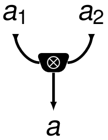
$\mathcal{H}$ net $\triangleq$ module + target–surjective



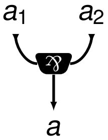
$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$



$\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$



$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$



$\langle a_1, a_2 \triangleright_{\parr} a \rangle$

$\langle a \triangleright_{\forall} b \rangle$

$\langle a \triangleright_{\exists} b \rangle$

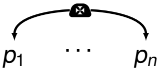
In an Untyped Setting this is not satisfying

V – Nets for MLL_2

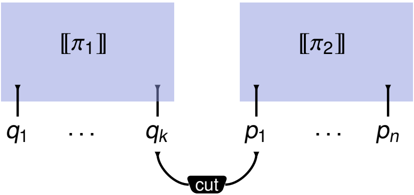
Proof nets: translating proofs to untyped nets

Translation (1/ 3)

$$\frac{}{A_1, \ldots, A_n}\ \maltese \qquad \xmapsto{[\![\cdot]\!]}$$

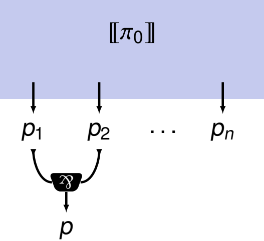


$$\frac{\begin{array}{c}\pi_1 \\ \| \\ \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2 \\ \| \\ A^\perp, \Delta\end{array}}{\Gamma, \Delta}\ \text{cut} \qquad \xmapsto{[\![\cdot]\!]}$$

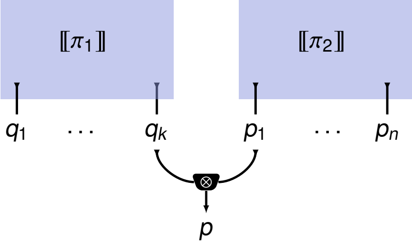


Translation (2/ 3)

$$\dfrac{\begin{array}{c}\pi_0 \\ \vdots \\ A, B, \Gamma\end{array}}{A \parr B, \Gamma}\ \parr \quad \xmapsto{[\![\cdot]\!]}$$

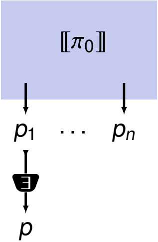


$$\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2 \\ \vdots \\ B, \Delta\end{array}}{\Gamma, A \otimes B, \Delta}\ \otimes \quad \xmapsto{[\![\cdot]\!]}$$

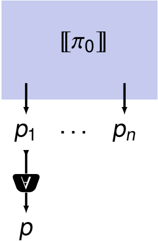


Translation (3/ 3)

$$\dfrac{\begin{array}{c}\pi_0 \, \Vert \\ A[X \leftarrow B], \Gamma\end{array}}{\exists X A, \Gamma}\,\exists \quad \xmapsto{[\![\cdot]\!]}$$



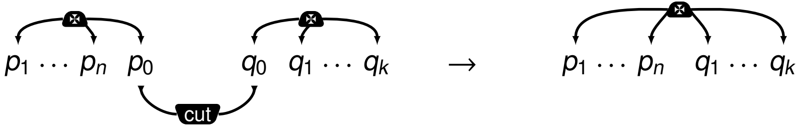
$$\dfrac{\begin{array}{c}\pi_0 \, \Vert \\ A[X \leftarrow Z], \Gamma\end{array}}{\forall X A, \Gamma}\,\forall \quad \xmapsto{[\![\cdot]\!]}$$

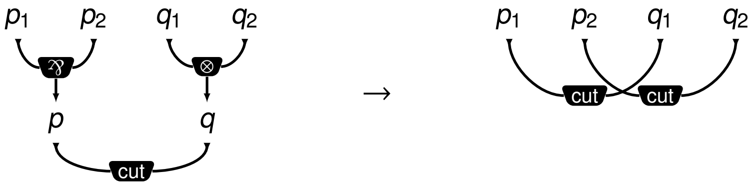


V – Nets for MLL_2

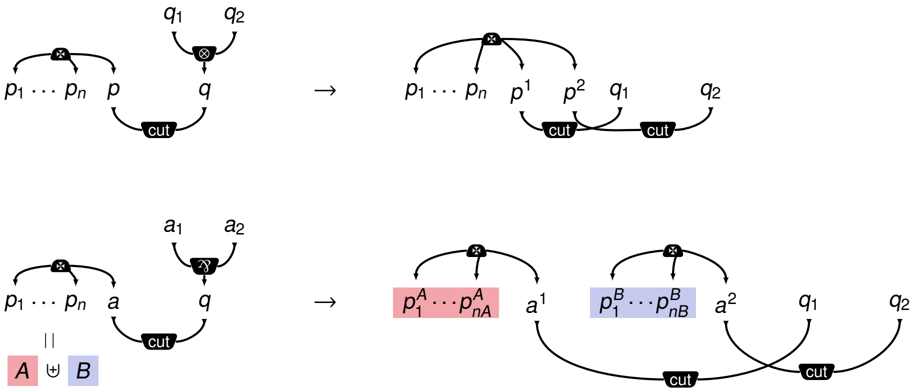
Cut elimination

Untyped Cut Elimination (1/3)

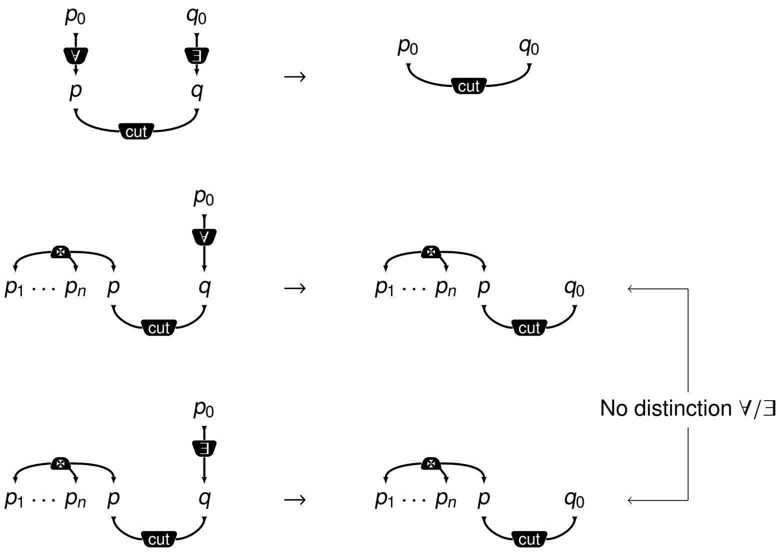




Untyped Cut Elimination (2/3)



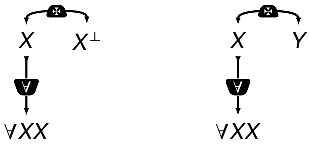
Untyped Cut Elimination (3/3)



V – Nets for MLL_2

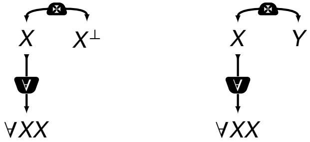
Limits to the naive untyped approach

Naive untyping (1/ 10)



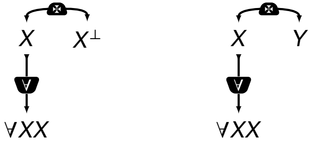
Naive untyping (2/ 10)

$$\left[\!\left[\ \dfrac{\dfrac{}{X, X^{\perp}}\,\maltese}{\forall X X, X^{\perp}}\,\forall\ \right]\!\right] =$$



Naive untyping (3/ 10)

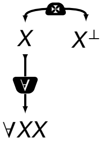
$$\left[\!\left[\; \cfrac{\cfrac{}{X, X^{\perp}}\ \parr}{\forall X X, X^{\perp}}\ \forall \;\right]\!\right] =$$

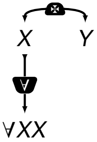


$$= \left[\!\left[\; \cfrac{\cfrac{}{X, Y}\ \parr}{\forall X X, Y}\ \forall \;\right]\!\right]$$

Naive untyping (4/ 10)

$$\left\llbracket \dfrac{\dfrac{}{X, X^{\perp}}\,\maltese}{\forall X X, X^{\perp}}\,\forall \right\rrbracket =$$



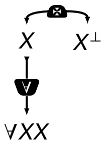


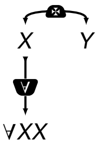
$$= \left\llbracket \dfrac{\dfrac{}{X, Y}\,\maltese}{\forall X X, Y}\,\forall \right\rrbracket$$

X occurs free in context
The ($\forall$)-rule is not valid

Naive untyping (5/ 10)

$$\left[\!\!\left[\ \dfrac{\dfrac{}{X, X^{\perp}}\ \maltese}{\forall X X, X^{\perp}}\ \forall\ \right]\!\!\right] =$$





$$= \left[\!\!\left[\ \dfrac{\dfrac{}{X, Y}\ \maltese}{\forall X X, Y}\ \forall\ \right]\!\!\right]$$

X occurs free in context
The ($\forall$)-rule is not valid

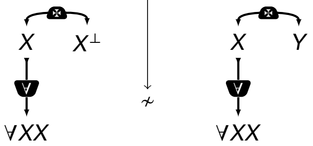
X doesn't occur free in context
The ($\forall$)-rule is valid

Naive untyping (6/ 10)

$$\left[\!\left[\dfrac{\overline{X, X^\perp}\;\maltese}{\forall X X, X^\perp}\;\forall \right]\!\right] =$$

X occurs free in context
The $(\forall)$-rule is not valid

Distinct w.r.t. cut-elim
$\{S_1\}^\perp \neq \{S_2\}^\perp$



$$= \left[\!\left[\dfrac{\overline{X, Y}\;\maltese}{\forall X X, Y}\;\forall \right]\!\right]$$

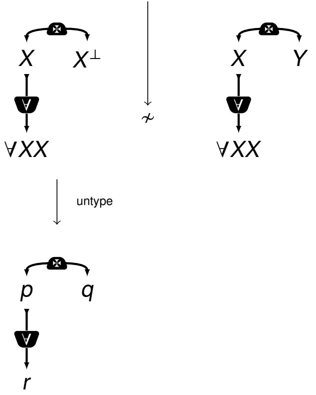
X doesn't occur free in context
The $(\forall)$-rule is valid

Naive untyping (7/ 10)

Distinct w.r.t. cut-elim

$\{S_1\}^\perp \neq \{S_2\}^\perp$

$$\left[\!\left[\frac{\overline{X, X^\perp}\ \maltese}{\forall X X, X^\perp}\ \forall \right]\!\right] =$$



$$= \left[\!\left[\frac{\overline{X, Y}\ \maltese}{\forall X X, Y}\ \forall \right]\!\right]$$

X occurs free in context
The ($\forall$)-rule is not valid

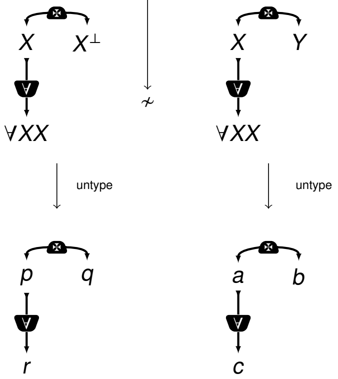
X doesn't occur free in context
The ($\forall$)-rule is valid

Naive untyping (8/ 10)

Distinct w.r.t. cut-elim

$\{S_1\}^\perp \neq \{S_2\}^\perp$

$$\left[\!\!\left[\ \dfrac{\overline{X, X^\perp}\ \maltese}{\forall X X, X^\perp}\ \forall\ \right]\!\!\right] =$$

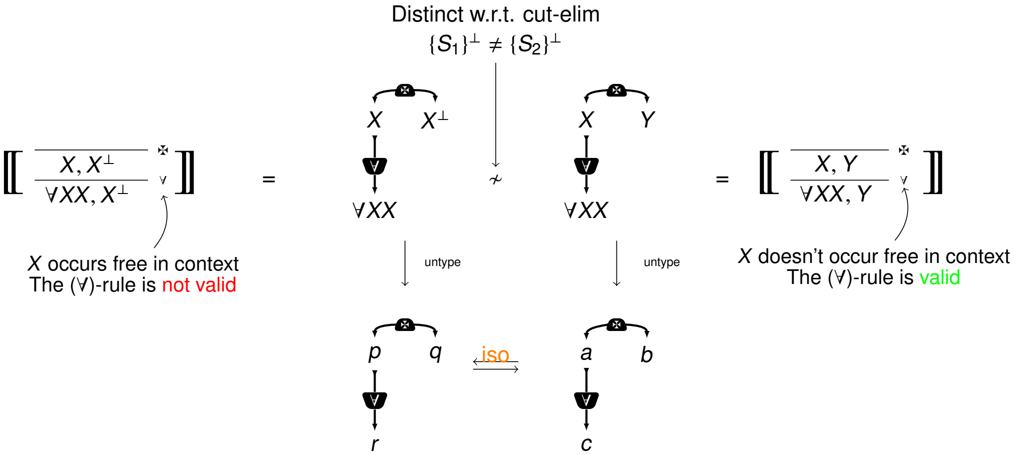


$$= \left[\!\!\left[\ \dfrac{\overline{X, Y}\ \maltese}{\forall X X, Y}\ \forall\ \right]\!\!\right]$$

X occurs free in context
The $(\forall)$-rule is not valid

X doesn't occur free in context
The $(\forall)$-rule is valid

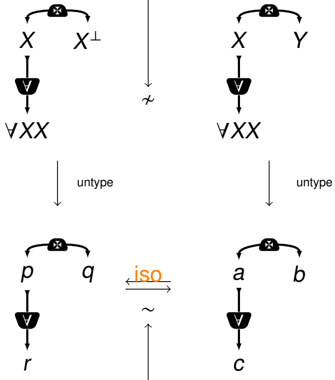
Naive untyping (9/ 10)



Naive untyping (10/ 10)

Distinct w.r.t. cut-elim
$\{S_1\}^\perp \neq \{S_2\}^\perp$

$$\left[\!\!\left[\dfrac{\overline{\quad X, X^\perp \quad}\ \parr}{\forall X X, X^\perp}\ \forall \right]\!\!\right] =$$



$$= \left[\!\!\left[\dfrac{\overline{\quad X, Y \quad}\ \parr}{\forall X X, Y}\ \forall \right]\!\!\right]$$

X occurs free in context
The $(\forall)$-rule is not valid

X doesn't occur free in context
The $(\forall)$-rule is valid

Similar w.r.t. cut-elim
$\{S_1\}^\perp = \{S_2\}^\perp$

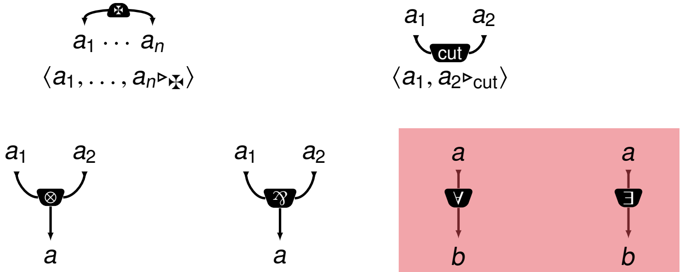
VI – Typed nets for MLL_2

Generating the set of nets

Typed Nets (1/ 4)

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

$\mathcal{H}$ net $\triangleq$ module + target–surjective



$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$ $\quad$ $\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$

$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$ $\quad$ $\langle a_1, a_2 \triangleright_{⅋} a \rangle$ $\quad$ $\langle a \triangleright_{\forall} b \rangle$ $\quad$ $\langle a \triangleright_{\exists} b \rangle$

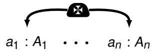
typed hypergraph $\mathcal{H}$
$\triangleq$
$\mathcal{H}$+ typing $\tau : V \to \mathcal{F}_{\mathrm{MLL2}}$

In an Untyped Setting this is not satisfying

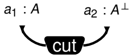
Typed Nets (2/ 4)

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

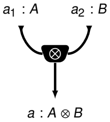
$\mathcal{H}$ net $\triangleq$ module + target–surjective



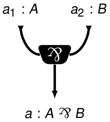
$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$



$\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$



$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$



$\langle a_1, a_2 \triangleright_{\parr} a \rangle$

$\langle a \triangleright_{\forall} b \rangle$

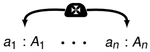
$\langle a \triangleright_{\exists} b \rangle$

typed hypergraph $\mathcal{H}$
$\triangleq$
$\mathcal{H}$+ typing $\tau : V \to \mathcal{F}_{\mathrm{MLL2}}$

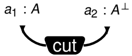
In an Untyped Setting this is not satisfying

Typed Nets (3/ 4)

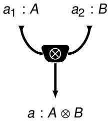
$\mathcal{H}$ module $\triangleq$ modular and sum of the links below
$\mathcal{H}$ net $\triangleq$ module + target–surjective



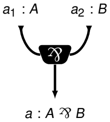
$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$



$\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$

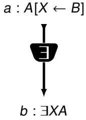


$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$



$\langle a_1, a_2 \triangleright_{\parr} a \rangle$

$\langle a \triangleright_{\forall} b \rangle$



$\langle a \triangleright_{\exists} b \rangle$

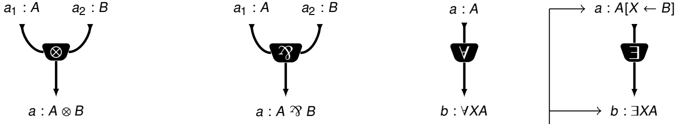
typed hypergraph $\mathcal{H}$
$\triangleq$
$\mathcal{H}$+ typing $\tau : V \to \mathcal{F}_{\mathrm{MLL2}}$

Typed Nets (4/ 4)

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

$\mathcal{H}$ net $\triangleq$ module + target–surjective

$\langle a_1, \ldots, a_n \triangleright_{\maltese} \rangle$ $\qquad$ $\langle a_1, a_2 \triangleright_{\mathrm{cut}} \rangle$



$\langle a_1, a_2 \triangleright_{\otimes} a \rangle$ $\qquad$ $\langle a_1, a_2 \triangleright_{\parr} a \rangle$ $\qquad$ $\langle a \triangleright_{\forall} b \rangle$ $\qquad$ $\langle a \triangleright_{\exists} b \rangle$

typed hypergraph $\mathcal{H}$
$\triangleq$
$\mathcal{H}$+ typing $\tau : V \to \mathcal{F}_{\mathrm{MLL2}}$

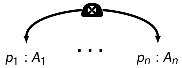
Infinitely many formulas are in such relation!

VI – Typed nets for MLL_2

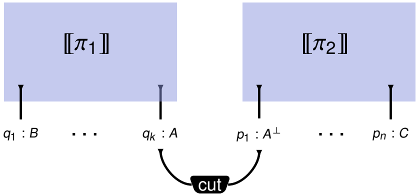
Proof nets

Typed Translation (1/ 3)

$$\frac{}{A_1, \ldots, A_n}\;\maltese \qquad \overset{[\![\cdot]\!]}{\longmapsto}$$

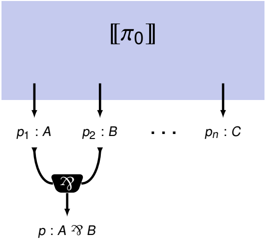


$$\frac{\begin{array}{c}\pi_1 \\ \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2 \\ A^{\perp}, \Delta\end{array}}{\Gamma, \Delta}\;\text{cut} \qquad \overset{[\![\cdot]\!]}{\longmapsto}$$

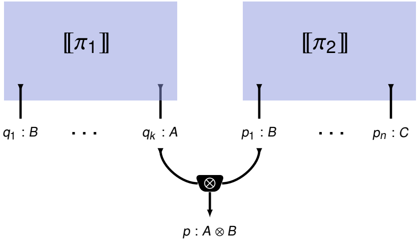


Typed Translation (2/ 3)

$$\dfrac{\begin{array}{c}\pi_0 \\ \| \\ A, B, \Gamma\end{array}}{A \parr B, \Gamma}\ \parr \quad \xmapsto{[\![\cdot]\!]}$$

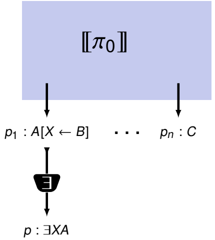


$$\dfrac{\begin{array}{c}\pi_1 \\ \| \\ \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2 \\ \| \\ B, \Delta\end{array}}{\Gamma, A \otimes B, \Delta}\ \otimes \quad \xmapsto{[\![\cdot]\!]}$$

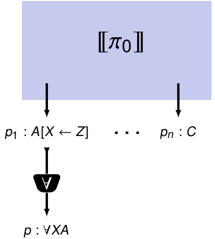


Typed Translation (3/ 3)

$$\dfrac{\begin{array}{c}\pi_0 \\ \Vert \\ A[X \leftarrow B], \Gamma\end{array}}{\exists XA, \Gamma}\ \exists \quad \overset{[\![\cdot]\!]}{\longmapsto}$$



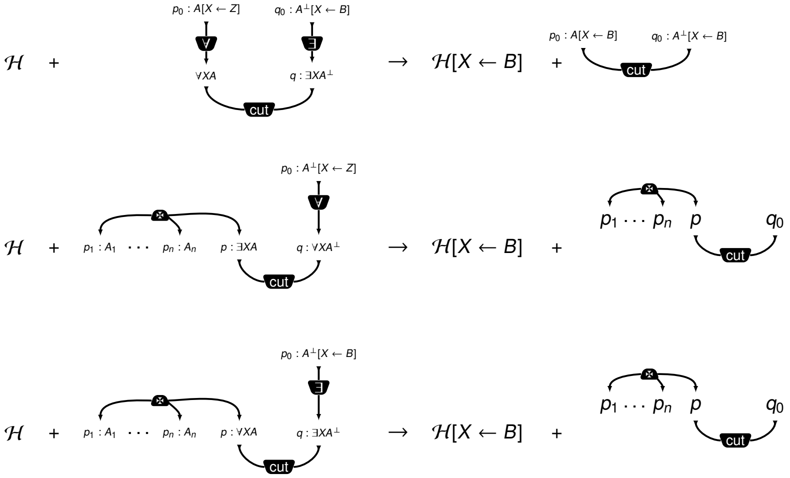
$$\dfrac{\begin{array}{c}\pi_0 \\ \Vert \\ A[X \leftarrow Z], \Gamma\end{array}}{\forall XA, \Gamma}\ \forall \quad \overset{[\![\cdot]\!]}{\longmapsto}$$



VI – Typed nets for MLL_2

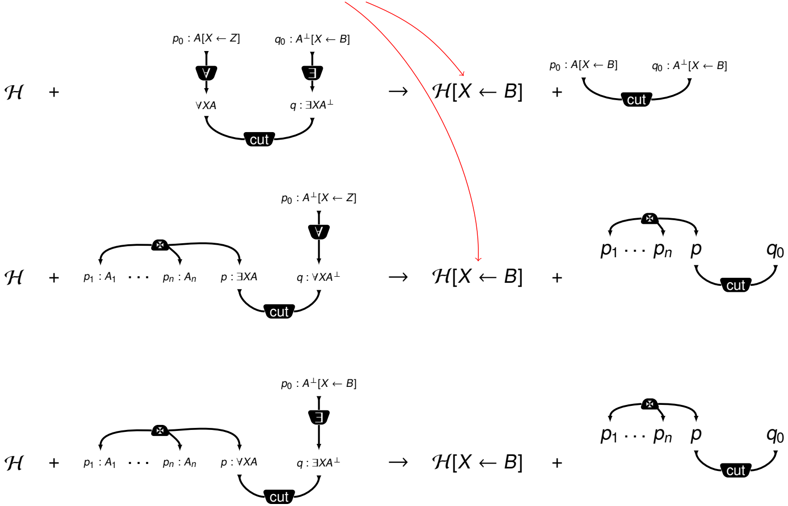
Cut elimination

Typed cut elimination (1/ 3)



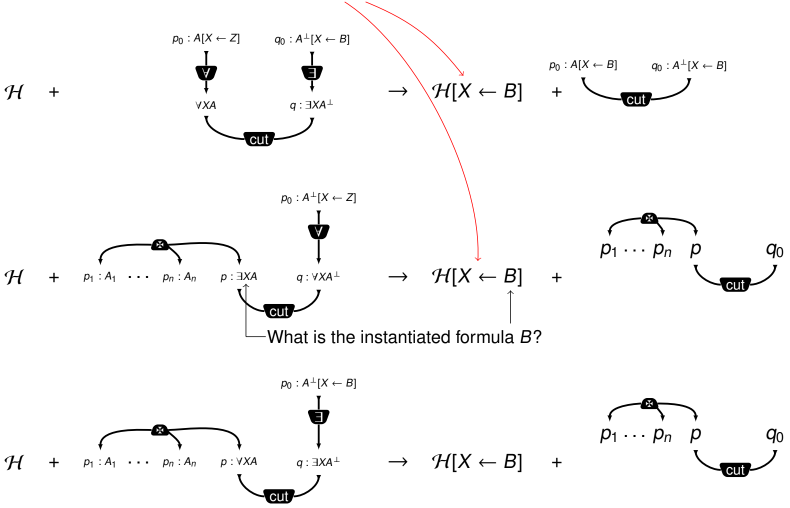
Typed cut elimination (2/ 3)

Expensive in time!
The rules are not local anymore!



Typed cut elimination (3/ 3)

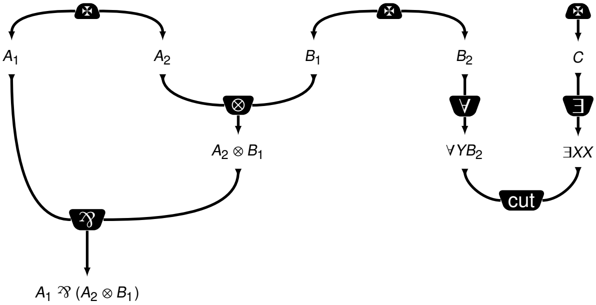
Expensive in time!
The rules are not local anymore!



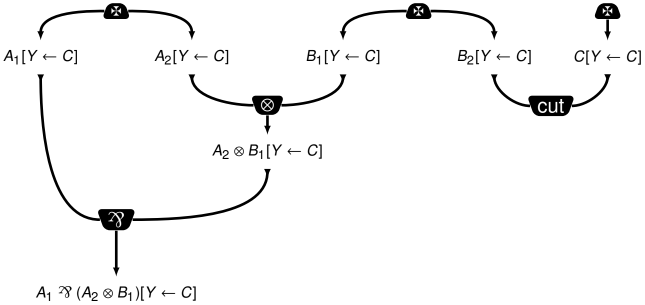
VI – Typed nets for MLL_2

The complexity of typed cut elimination: non–locality

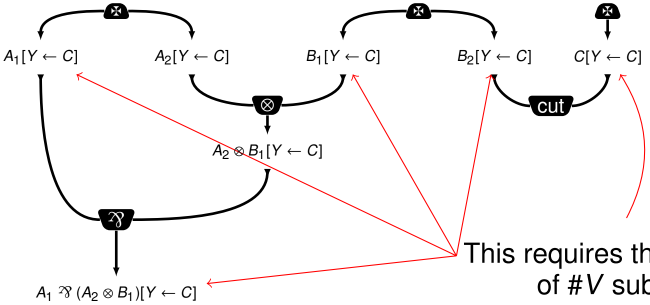
Typed cut–elimination is not local (1/ 3)



Typed cut–elimination is not local (2/ 3)



Typed cut–elimination is not local (3/ 3)



This requires the computation of $\#V$ substitutions!

VI – Typed nets for MLL_2

The complexity of typed cut elimination: iterated substitutions

Complexity of substitution (1/ 8)

$$\#S = \text{number of links in the net } S$$
$$\#_X A = \text{number of occurence of the p.v. } X \text{ in the formula } A$$

Proposition. (in MLL)

$$S \to S' \Rightarrow \#S > \#S'$$

Complexity of substitution (2/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$$S \to S' \Rightarrow \#S > \#S'$$

$\Rightarrow$ MLL normalization takes linear time

Complexity of substitution (3/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL) $S \to S' \Rightarrow \#S > \#S'$	$\Rightarrow$	MLL normalization takes linear time		**Proposition.** (in MLL2) $S \to S' \Rightarrow \#S > \#S'$

Complexity of substitution (4/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$S \to S' \Rightarrow \#S > \#S'$

$\Rightarrow$ MLL normalization takes linear time

Proposition. (in MLL2)

$S \to S' \Rightarrow \#S > \#S'$

$\not\Rightarrow$ MLL2 normalization takes linear time

Complexity of substitution (5/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$S \to S' \Rightarrow \#S > \#S'$

$\Rightarrow$ MLL normalization takes linear time

Proposition. (in MLL2)

$S \to S' \Rightarrow \#S > \#S'$

$\not\Rightarrow$ MLL2 normalization takes linear time

Why?

Complexity of substitution (6/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$S \to S' \Rightarrow \#S > \#S'$

$\Rightarrow$ MLL normalization takes linear time

Proposition. (in MLL2)

$S \to S' \Rightarrow \#S > \#S'$

$\not\Rightarrow$ MLL2 normalization takes linear time

Why?

in MLL2 types need to be computed during cut elimination

Complexity of substitution (7/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$S \to S' \Rightarrow \#S > \#S'$

$\Rightarrow$ MLL normalization takes linear time

Proposition. (in MLL2)

$S \to S' \Rightarrow \#S > \#S'$

$\not\Rightarrow$ MLL2 normalization takes linear time

Why?

in MLL2 types need to be computed during cut elimination

Proposition.

$\#_Y(A[X \leftarrow B]) \geq \#_X(A) * \#_Y(B)$

Complexity of substitution (8/ 8)

$\#S$ = number of links in the net S
$\#_X A$ = number of occurence of the p.v. X in the formula A

Proposition. (in MLL)

$S \to S' \Rightarrow \#S > \#S'$

$\Rightarrow$ MLL normalization takes linear time

Proposition. (in MLL2)

$S \to S' \Rightarrow \#S > \#S'$

$\not\Rightarrow$ MLL2 normalization takes linear time

Why?

in MLL2 types need to be computed during cut elimination

Proposition.

$\#_Y(A[X \leftarrow B]) \geq \#_X(A) * \#_Y(B)$

Iterating substitutions $\Rightarrow$ Iterate multiplications

computing types is of exponential complexity

VII – Nets with pointers for MLL_2

Nets with pointers

Nets with pointers (1/ 8)

pointers $\triangleq$ $\langle a \to_{\forall} b \rangle$ $\langle a \to_{\exists} b \rangle$ $\langle a \; \langle a \to_{\forall} b \rangle \; b \rangle$

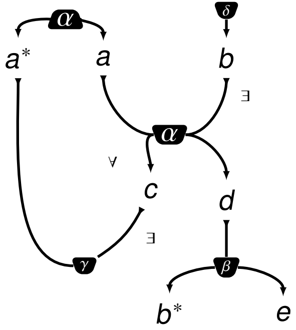
Nets with pointers (2/ 8)

pointers $\triangleq$ $\langle a \to_\forall b\rangle$ $\langle a \to_\exists b\rangle$ $\langle a\ \cancel{\langle a \to_\forall b\rangle}\ b\rangle$

Net with pointers $\triangleq$ Net

a net $\mathcal{H}$

$$\langle a, b \triangleright_\alpha c, d\rangle$$
$$+$$
$$\langle d \triangleright_\beta b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \rangle$$
$$+$$
$$\langle \triangleright_\delta b\rangle$$
$$+$$
$$\langle \triangleright_\delta a, a^*\rangle$$



Nets with pointers (3/ 8)

pointers $\triangleq$ $\langle a \rightarrow_\forall b \rangle$ $\langle a \rightarrow_\exists b \rangle$ $\langle a \, \langle a_\forall b \rangle_\exists \, b \rangle$

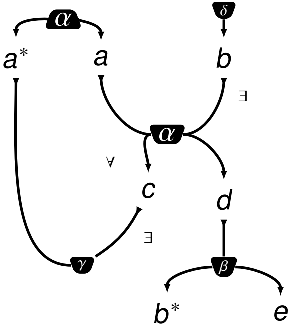
Net with pointers $\triangleq$ Net + pointers

a net $\mathcal{H}$

$\langle a, b \triangleright_\alpha c, d \rangle$
$+$
$\langle d \triangleright_\beta b^*, e \rangle$
$+$
$\langle a^*, c \triangleright_\gamma \rangle$
$+$
$\langle \triangleright_\delta b \rangle$
$+$
$\langle \triangleright_\delta a, a^* \rangle$

+

pointers **P**



Nets with pointers (4/ 8)

pointers $\triangleq$ $\langle a \rightarrow_\forall b\rangle$ $\langle a \rightarrow_\exists b\rangle$ $\langle a \leftarrow_\forall b\rangle$

Net with pointers $\triangleq$ Net + pointers

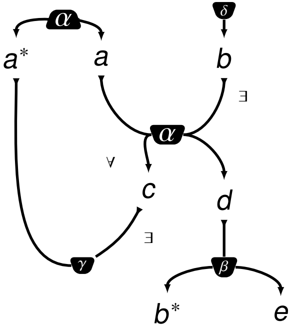
$\uparrow$
target–disjoint
source–disjoint

a net $\mathcal{H}$

$$\langle a, b \triangleright_\alpha c, d\rangle$$
$$+$$
$$\langle d \triangleright_\beta b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \rangle$$
$$+$$
$$\langle \triangleright_\delta b\rangle$$
$$+$$
$$\langle \triangleright_\delta a, a^*\rangle$$

+

pointers **P**



Nets with pointers (5/ 8)

pointers $\triangleq$ $\langle a \to_\forall b\rangle$ $\langle a \to_\exists b\rangle$ $\langle a \; \langle a \to_\forall b\rangle_\forall \; b\rangle$

Net with pointers $\triangleq$ Net + pointers

Net: target–disjoint, source–disjoint

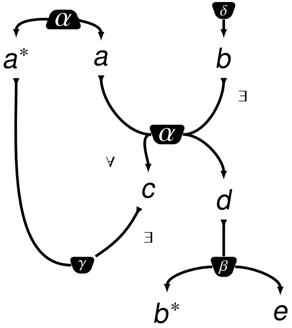
pointers: no restrictions

a net $\mathcal{H}$

$$\langle a, b \triangleright_\alpha c, d\rangle$$
$$+$$
$$\langle d \triangleright_\beta b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \rangle$$
$$+$$
$$\langle \triangleright_\delta b\rangle$$
$$+$$
$$\langle \triangleright_\delta a, a^*\rangle$$

+

pointers P



Nets with pointers (6/ 8)

pointers $\triangleq$ $\langle a \to_\forall b\rangle$ $\langle a \to_\exists b\rangle$ $\langle a \ \langle a_\forall \ b\rangle_\forall \ b\rangle$

Net with pointers $\triangleq$ Net + pointers

Net: target–disjoint, source–disjoint

pointers: no restrictions

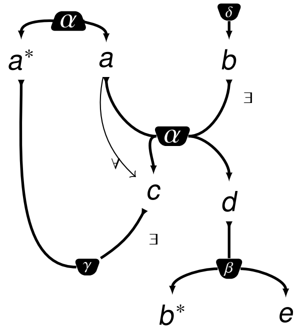
a net $\mathcal{H}$

$$\langle a, b \ \triangleright_\alpha \ c, d\rangle$$
$$+$$
$$\langle d \ \triangleright_\beta \ b^*, e\rangle$$
$$+$$
$$\langle a^*, c \triangleright_\gamma \ \rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ b\rangle$$
$$+$$
$$\langle \ \triangleright_\delta \ a, a^*\rangle$$

+

pointers P

$$\langle a \to_\forall c\rangle$$



Nets with pointers (7/ 8)

pointers $\triangleq$ $\langle a \to_\forall b\rangle$ $\langle a \to_\exists b\rangle$ $\langle a \langle a \to_\forall b\rangle_\forall b\rangle$

Net with pointers $\triangleq$ Net + pointers

target–disjoint
source–disjoint

no restrictions

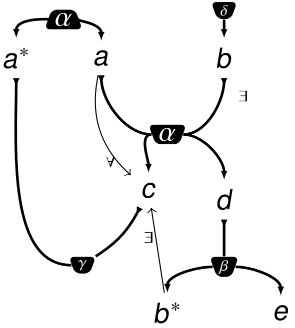
a net $\mathcal{H}$

$$\langle a, b \triangleright_\alpha c, d\rangle + \langle d \triangleright_\beta b^*, e\rangle + \langle a^*, c \triangleright_\gamma \rangle + \langle \triangleright_\delta b\rangle + \langle \triangleright_\delta a, a^*\rangle$$

+

pointers **P**

$$\langle a \to_\forall c\rangle + \langle b^* \to_\exists c\rangle$$



Nets with pointers (8/ 8)

pointers $\triangleq$ $\langle a \rightarrow_\forall b\rangle$ $\langle a \rightarrow_\exists b\rangle$ $\langle a\ \langle a \rightarrow_\forall b\rangle_\forall\ b\rangle$

Net with pointers $\triangleq$ Net + pointers

Net: target–disjoint, source–disjoint

pointers: no restrictions

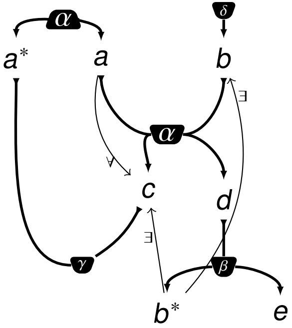
a net $\mathcal{H}$

$$\langle a, b \triangleright_\alpha c, d\rangle + \langle d \triangleright_\beta b^*, e\rangle + \langle a^*, c \triangleright_\gamma \rangle + \langle \triangleright_\delta b\rangle + \langle \triangleright_\delta a, a^*\rangle$$

\+

pointers P

$$\langle a \rightarrow_\forall c\rangle + \langle b^* \rightarrow_\exists c\rangle + \langle b^* \rightarrow_\exists b\rangle$$

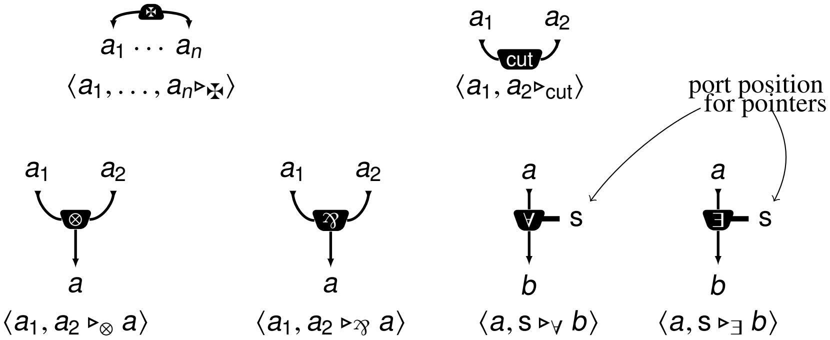


Links for MLL_2 nets

$\mathcal{H}$ module $\triangleq$ modular and sum of the links below

$\mathcal{H}$ net $\triangleq$ module + target–surjective

$\mathcal{H}$ net with pointers $\triangleq$ net + pointers link



VII – Nets with pointers for MLL_2

Localized sequent calculus

Localized sequent calculus

$$\frac{}{\mathbf{x_1} \cdot A_1, \ldots, \mathbf{x_n} \cdot A_n} \maltese \qquad \frac{\mathbf{x}A^{\perp}, \Gamma \quad \mathbf{y}A, \Delta}{\Gamma, \Delta} \text{cut}$$

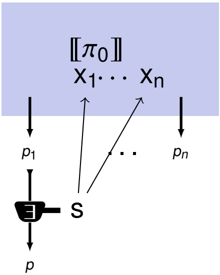
$$\frac{\mathbf{x} \cdot A, \mathbf{y} \cdot B, \Gamma}{\mathbf{p} \cdot A \parr B, \Gamma} \parr \qquad \frac{\mathbf{x}A, \Gamma \quad \mathbf{y}B, \Delta}{\mathbf{p}A \otimes B, \Gamma, \Delta} \otimes$$

$$\frac{\Gamma, \mathbf{p}A}{\Gamma, \mathbf{q}\forall X \mathbf{p}A} \forall \qquad \frac{\Gamma, \mathbf{p}A[\mathbf{x}_1, \ldots, \mathbf{x}_n \leftarrow B]}{\Gamma, \mathbf{q}\exists X \mathbf{p}A[\mathbf{x}_1, \ldots, \mathbf{x}_n \leftarrow X]} \forall$$

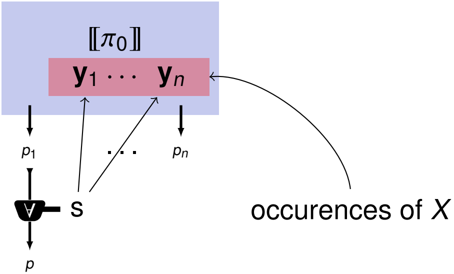
VII – Nets with pointers for MLL_2

Proof nets

$$\dfrac{\begin{array}{c}\pi_0 \\ \Vert \\ \mathbf{p}_1 A, \Gamma\end{array}}{\mathbf{p}\exists X \mathbf{p}_1 A[\mathbf{x}_1, \ldots, \mathbf{x}_n \leftarrow X], \Gamma}\ \exists \qquad \xmapsto{[\![\cdot]\!]}$$



$$\dfrac{\begin{array}{c}\pi_0 \\ \Vert \\ \mathbf{p}_1 A, \Gamma\end{array}}{\mathbf{p}\forall X \mathbf{p}_1 A, \Gamma}\ \forall \qquad \xmapsto{[\![\cdot]\!]}$$

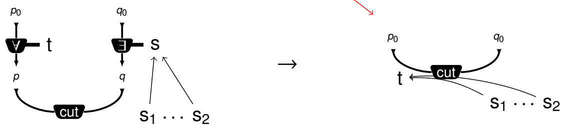


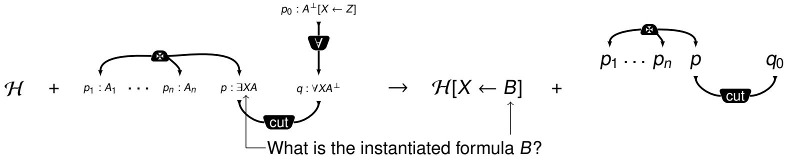
VII – Nets with pointers for MLL_2

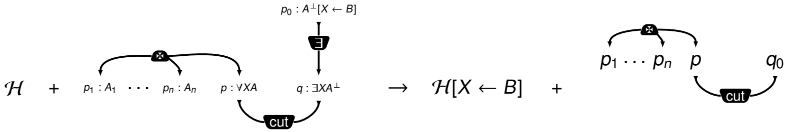
Cut elimination

Cut elimination with pointers (1/ 6)

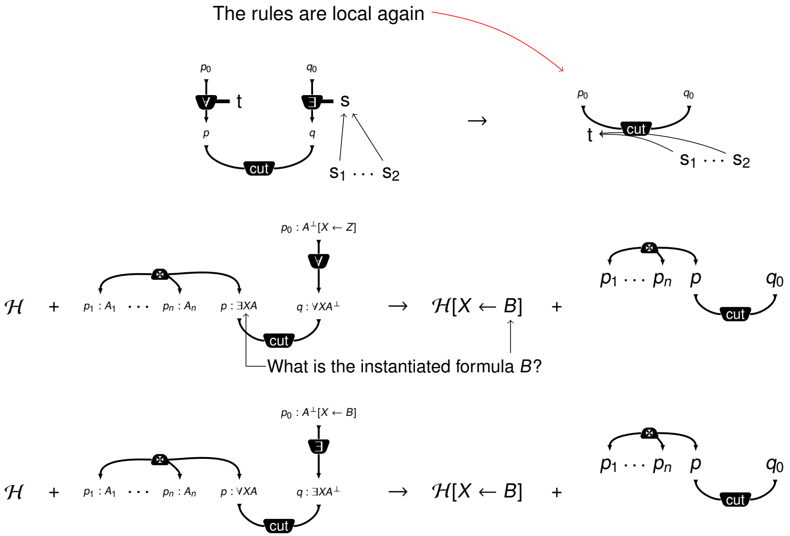
The rules are local again



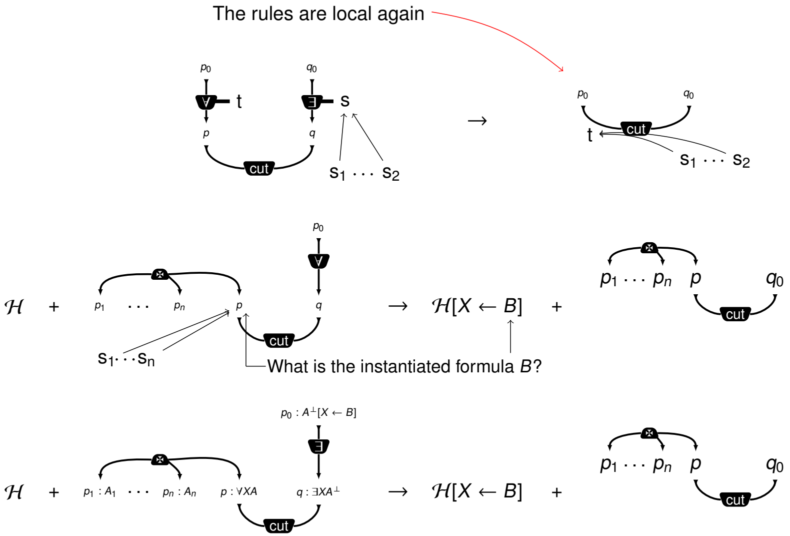




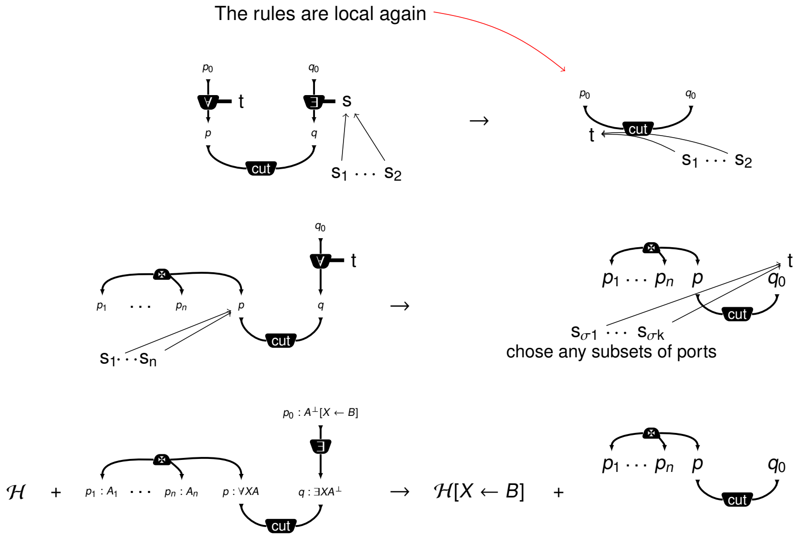
Cut elimination with pointers (2/ 6)



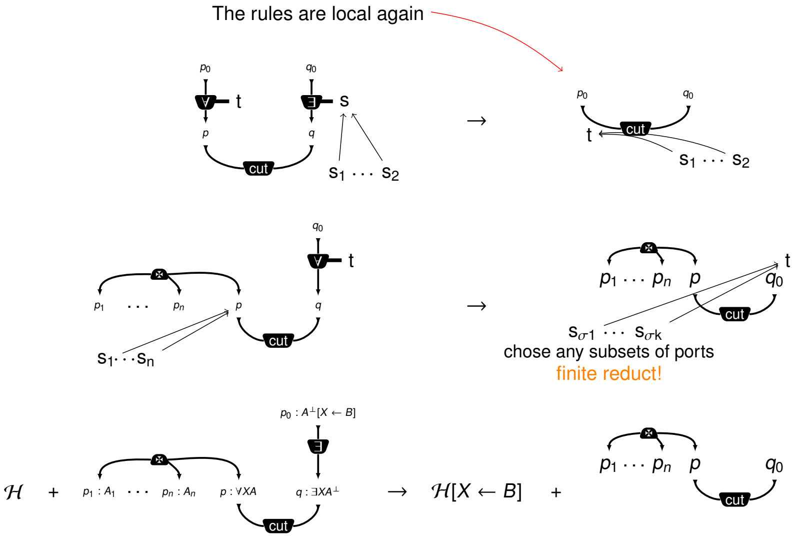
Cut elimination with pointers (3/ 6)



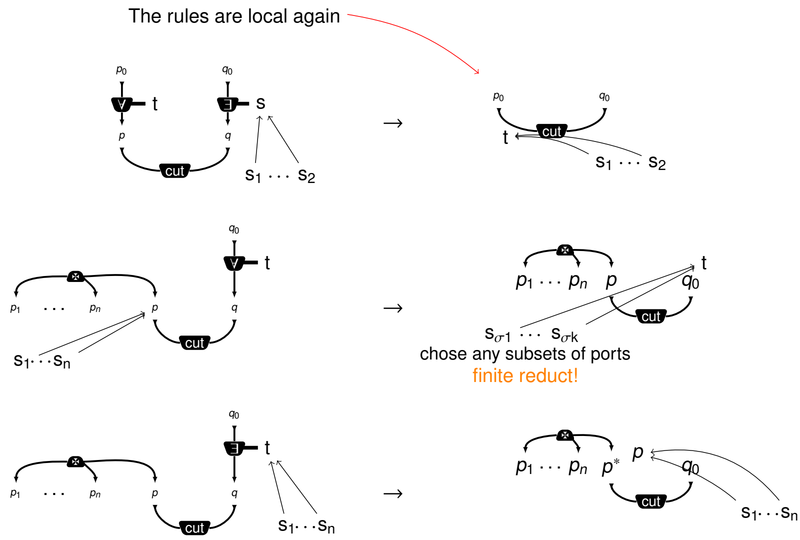
Cut elimination with pointers (4/ 6)



Cut elimination with pointers (5/ 6)

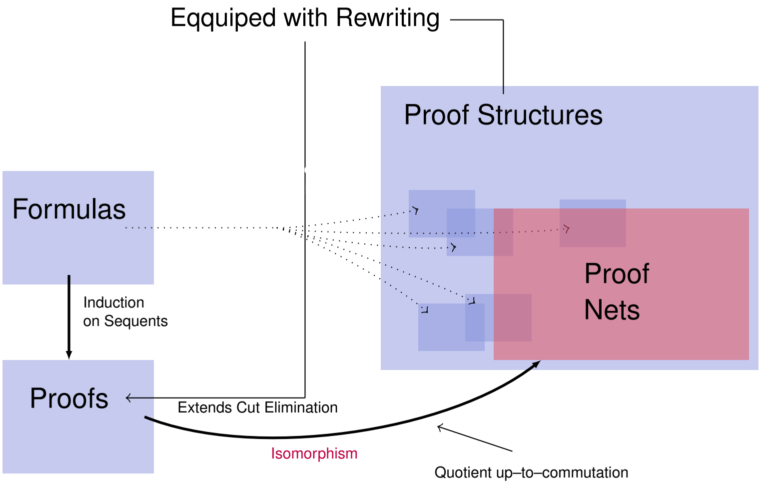


Cut elimination with pointers (6/ 6)

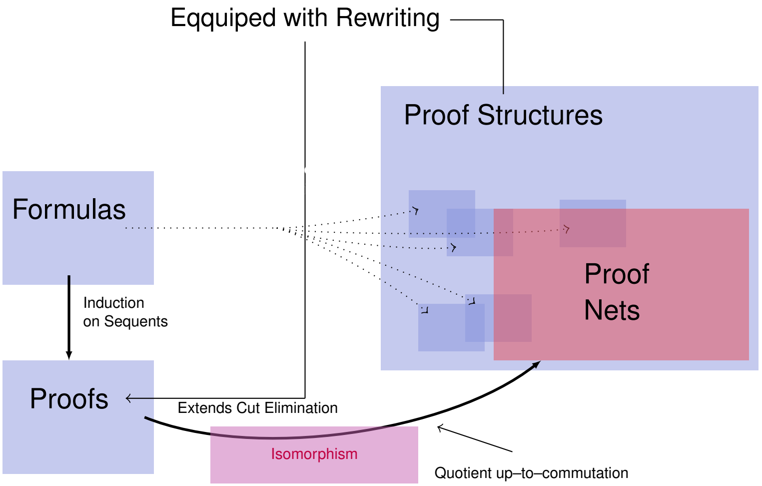


Conclusion

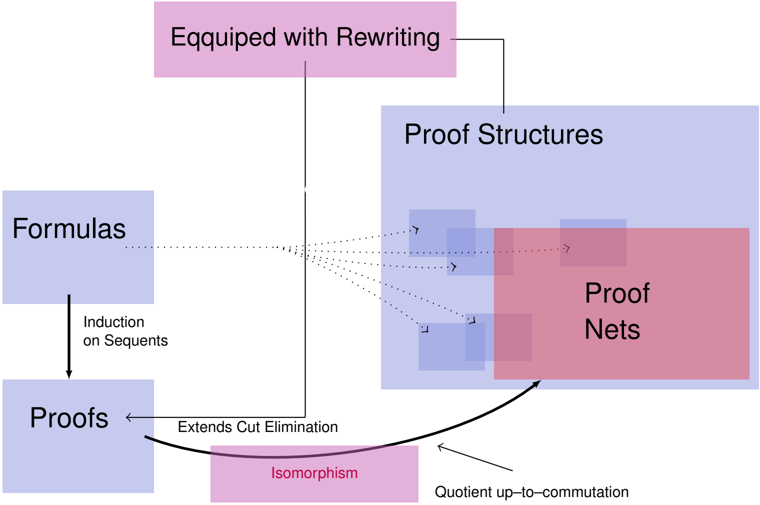
Conclusion (1/ 3)



Conclusion (2/ 3)



Conclusion (3/ 3)



Thank You