Categories of Games

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Abstract

We define basic notions coming from game semantics, specifically in the setting of Hyland–Ong games. We show that composition of strategies is well defined and associative, and that the copycat strategy behave as a neutral element with respect to composition, thus exhibiting a category of games. We then exhibit other categories of games by studying well–known properties of strategies that are compatible with composition (winning, innocence, well bracketing, visibility).

Although some of our proofs might defer from the currently existing one in the literature, none of the results present in this note are novels. These notes are intended to be an introduction to game semantics and some of its core concepts.

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Introduction

Game semantics comes from Dialogical Logic a field of study introduced by Lorenzen in the beginning of the sixties [Lor60], which was further developed by his student Kuno Lorenz during his PhD thesis [Lor81]. Inspired by the game theory of Von Neumann [Neu28], this novel approach seeked to give a pragmatic conception of "logical" *truth*, that would no longer be based on the informal use of logical reasoning in the meta–languange [Lor81]. In Dialogical Logic the notion of truth is defined around the notion of interaction or dialogue between two agents, the *player* and the *opponent*. For a given formula A the player seeks to prove A whereas the opponent seeks to disprove it, we would say that the formula A is disputed. Morally, when a formula A is disputed, if the player can defend his claims from every "attack" of the opponent, and if he can do so until the possibilities of the opponent are exhausted, the formula A is true (and it would be false otherwise). The description of how the player must defend from the opponent is called a *strategy* and represents how the player must play the during the game or dialogue – it could for instance be a partial function mapping opponent moves to player moves. A strategy for a formula A is then *winning* if whenever the player follow it the formula A is proven. In practice, dialogical logic has a constructive nature due to the so called *definability theorems* stating that from a winning strategy of A one can extract the proof of A in the sequent calculus. Reciprocally, each proof of A in the sequent calculus yields a winning strategy for A.

On the other hand, denotational semantics was initiated in the seventies by Dana Scott and Christopher Stratchey, and was introduced in order to provide a meaning for computer programs. The method proposed by denotational semantics consist of associating to a program P a *mathematical object* called its denotation $[\![P]\!]$, the associated mathematical structures could be continuous functions between so–called *Scott domains* [Sco72], stable functions between coherent spaces [Gir86], strategies [HO00] etc.... Denotational semantics yielded a tool for reasoning about the equivalences of programs, a problem that was known to be difficult (in fact undecidable): the usual notion of program equivalence is called *observational equivalence* and two program are P_1 and P_2 are equivalent $P_1 \simeq P_2$ whenever for any *context* $C\langle\cdot\rangle$, $C\langle P_1\rangle$ and $C\langle P_2\rangle$ reduces to the same normal form (or both diverge). Indeed such an equivalence is difficult to decide because of the universal quantification over the contexts $C\langle\cdot\rangle$. Denotational semantics equates term up to the reduction steps of the computation model, for intance in the lambda–calculus that would mean that whenever $t \to_\beta u$ we have the equality of the denotations $[\![t]\!] = [\![u]\!]$. In a sense, the denotations flatten the space of programs at least along reduction. This flattening of the program space naturally yields a notion of equivalence where two programs P_1 and P_2 are equivalent whenever they have the same denotation i.e. $[\![P_1]\!] = [\![P_2]\!]$. A denotational semantics is then said to be *fully abstract* whenever it equates the programs that are observationally equivalent. Thus a fully abstract denotational semantics provide a tool for reasoning about observational equivalence.

Denotational semantics has been studied for the functional programming language PCF, Gordon Plotkin showed in the 1977 that the language PCF extended with a "parallel–or" was fully abstract for the scott domains semantics [Plo99]. Yet the question of whether there existed a fully abstract semantics for PCF remained open for several years. The answer to this question – due to Martin Hyland and Luke Ong [HO00] – was given 2000 and turned out to be positive. The semantics provided by Hyland and Ong was based on the games and dialogues coming from the dialogical logic, it was then named *game semantics* and the game involved in the work of Hylang–Ong are called *Hyland–Ong games*, they are considered to be *sequential games*. This results drew attention to the expressiveness of game semantics, and full–abstraction results where obtained for other programming languages aswell, sustaining the idea that game semantics is a powerful tool to reason about programming languages: Abramsky and McCusker obtained full abstraction for the lazy λ–calculus [AM95], and for Idealized Algol (denoted IA) [AM99], Lair obtained a full abstraction result for PCF + CONTROL [Lai97]. Full–abstraction by the means of games semantics was also obtained for concurrent languages such as the π–calculus by Fiore and Sangiorgi [FMS02].

Game semantics proved to be closely related to the paradigm of the Geometry of Interaction (GOI) introduced by Jean–Yves Girard in 1989 [Gir89]. In the case of multiplicative Linear Logic the trips on a proof structures can be related to play on the game associated to the proof structure [Bai99]. In the general case, GOI generates a game semantics in the sense that the permutations involved in the Geometry of Interaction can be seen as really simple plays, enriching these plays with more structure (such as determinism, innocence, well bracketing ...) leads to Game semantics.

In this note we show that the collection of Hyland–Ong games forms a category, in which the objects are games and the morphisms correspond to strategies. Namely this equates to showing that strategies (in the sense of Hyland–Ong) are stable under composition and have a neutral element: the *copycat strategy* [Cla22]. Then we show how this is still true for visibility, well–bracketing and innocence

[Har04] [Cla10], thus obtaining three categories. The method we choose to show these results relies on a notion of state diagrams for pointed strings (i.e. plays) this fruitful approach was introduced by Russ Harmer to show the stability under composition of innocent strategies [Har04]. Let us be clear on the fact that none of the results here are novels – although some proofs might defer from the existing one, for instance, we explicit the notion of state diagram and the proof of the propositions related to it. These notes are intended to be an introduction to game semantics and some of its core concepts.

1 General notions for games semantics

1.1 Moves, arenas and plays

From now on, we assume that we are given a set Σ of elements called atoms. We will denoted by $\mathbb{N}^*$ the free monoid on the set $\mathbb{N}$ of natural numbers. A element of $\mathbb{N}^*$ will be called an *address*. We denote by ε the empty sequence also called the *empty address*. The prefixed address ξ by an integer n is the concatenation $n \cdot \xi$. We will denote the concatenation of two addresses ξ_1 and ξ_2 as $\xi_1 \cdot \xi_2$.

Definition 1.1 (Move)**.** A *move* is a pair (ξ, a) where ξ is an address and a is an atom belonging to Σ. A move (ξ, a) will be denoted $\xi \cdot a$. Furthermore if there is no ambiguity we might not distinguish the element a from the move $\xi \cdot a$.

As of now we introduce two sets made of two element, the set *actors* $\{\mathsf{P}, \mathsf{O}\}$ made of the element P called the *player* and the element O called the *opponent*. Another set $\{\mathsf{q}, \mathsf{a}\}$ of elements called *dialectic roles* where q is called the *question polarity* and a the *answer polarity*.

Definition 1.2 (Actor and dialectic polarization)**.** Given a set of moves M an *actor polarization* of M is a function $\lambda^A : M \longrightarrow \{\mathsf{P}, \mathsf{O}\}$ which associates to each move an actor that is either the player or the opponent. A *dialectic polarization* of set of moves M is a function $\lambda^D : M \longrightarrow \{\mathsf{q}, \mathsf{a}\}$ that associates to each move its dialectic role, either an answer or a question. A *polarization* of a set of moves M is a function $\lambda : M \longrightarrow \{\mathsf{P}, \mathsf{O}\} \times \{\mathsf{q}, \mathsf{a}\}$ which associates to each move its actor and its dialectic role.

Definition 1.3 (Pre–Arena and games)**.** A *pre–arena* is a tuple $\mathcal{A} = (M_{\mathcal{A}}, \lambda_{\mathcal{A}}, \vdash_{\mathcal{A}}, I_{\mathcal{A}})$ where:

- $M_{\mathcal{A}}$ is a set of moves.
- $\lambda_{\mathcal{A}}$ is a polarization of $M_{\mathcal{A}}$.
- $\vdash_{\mathcal{A}}$ is a binary relation over the set of moves $M_{\mathcal{A}}$, called the *enabling relation*. A move m *enables* another move n if $m \vdash_{\mathcal{A}} n$.
- $I_{\mathcal{A}}$ is a subset of the set $M_{\mathcal{A}}$ of moves. If a move m belongs to $I_{\mathcal{A}}$ we say it is an *initial move*.

An *arena* or *game* is a pre–arena $\mathcal{A} = (M_{\mathcal{A}}, \lambda_{\mathcal{A}}, \vdash_{\mathcal{A}}, I_{\mathcal{A}})$ that is:

- *initial* if each initial move is an opponent question i.e. for any $m \in I_{\mathcal{A}}$ we have $\lambda_{\mathcal{A}}(m) = (\mathsf{O}, \mathsf{q})$.
- *alternated* if for each move m and n if m justifies n then the actor of the moves m and n must be distinct. Meaning that if $m \vdash_{\mathcal{A}} n$ then $\lambda^A_{\mathcal{A}}(m) \neq \lambda^A_{\mathcal{A}}(n)$.
- *argumentative* if for each move m and n if m enables n and n is an answer then m is a question. Meaning that if $m \vdash_{\mathcal{A}} n$ and $\lambda^D_{\mathcal{A}}(n) = \mathsf{a}$ then $\lambda^D_{\mathcal{A}}(m) = \mathsf{q}$.

The following arenas correspond to the type of the integers **Nat** and the type of booleans **Bool**.

$$\begin{tikzcd}[column sep=small] & & q \arrow[dll] \arrow[dl] \arrow[dr] & & \\ 0 & 1 & \cdots & n & \cdots \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & q \arrow[dl] \arrow[dr] & \\ \mathsf{tt} & & \mathsf{ff} \end{tikzcd}$$

Definition 1.4 (Address and relocalization of an arena)**.** Given M a set of moves we define the set of *addresses* of M by:

$$adr(M) = \{\xi \mid \xi \cdot a \in M\}.$$

A *relocalization* of M is a function $\kappa : adr(M) \longrightarrow \mathbb{N}^*$ that is injective and preserve the prefix order. The κ*–relocalization* (or recolazition under κ) of M is denoted $\kappa \cdot M$ and is defined to be:

$$\kappa \cdot M = \{\kappa(\xi) \cdot a \mid \xi \cdot a \in M\}.$$

The set of addresses of an arena $\mathcal{A}$ is set of addresses of its moves, meaning $adr(\mathcal{A}) = adr(M_{\mathcal{A}})$. A *relocalization* of an arena $\mathcal{A}$ is a relocalization of its set of moves $M_{\mathcal{A}}$. We define the relocalization of an arena $\mathcal{A}$ that we denote $\kappa\mathcal{A}$ as the following:

- The set of moves of the arena $\kappa \cdot \mathcal{A}$ is the relocalization under κ of the set of moves of $\mathcal{A}$, equationally: $M_{\kappa \cdot \mathcal{A}} = \kappa \cdot M_{\mathcal{A}}$.
- $\vdash_{\kappa \cdot \mathcal{A}} = \{\kappa(\xi_1) \cdot a_1, \kappa(\xi_2) \cdot a_2 \mid \xi_1 \cdot a_1 \vdash_{\mathcal{A}} \xi_2 a_2\}$.
- Given a move $\xi \cdot a$ of the arena $\kappa \cdot \mathcal{A}$ then its polarity is given by $\lambda_{\kappa \cdot \mathcal{A}}(\xi \cdot a) = \lambda_{\mathcal{A}}(\kappa^{-1}(\xi) \cdot a)$.
- The initial moves of $\kappa \cdot \mathcal{A}$ are the images of the initial moves of $\mathcal{A}$ under the relocalization κ, equationally: $I_{\kappa \cdot \mathcal{A}} = \kappa \cdot I_A$.

Definition 1.5 (pointed string, play and legal play)**.** A *play* or *word with pointer* or *pointed string* over a set of moves M, is a finite sequence of moves $s = (s_1, \dots, s_n)$ together with a partial function $p_s : \{s_1, \dots, s_n\} \to \{s_1, \dots, s_n\}$ called *pointer* such that if $p(s_j) = s_i$ then $i < j$. Given a pointed string s we say the move s_j *points to* or that it is *justified by* the move s_i if $p(s_j) = s_i$.

Given an arena A and s a pointed string of moves of A a question–move s_i occuring in s is *answered* if there exists a move s_j pointing to s_i such that s_j is an answer. Otherwise s_i is *unanswered* or *pending*.

Given a game $\mathcal{A}$ a play $s = (s_1, \dots, s_n)$ is:

- *Alternated* if any two consecutive moves s_i and s_{i+1} have a distinct actor, i.e. $\lambda^A_{\mathcal{A}}(s_i) \neq \lambda^A_{\mathcal{A}}(s_{i+1})$.
- *Well–opened* if any move s_i that points to no other move is an initial move.
- *Well–justified* if a move s_j points to a move s_i then s_i justifies s_j, $s_i \vdash_{\mathcal{A}} s_j$.
- *coherent* if whenever an answer s_j occurs in s justified by a move s_i then no unanswered question occur between s_i and s_j.

A *legal play* is a play that is alternated, well–opened and well–justified.

Definition 1.6 (restrictions)**.** Given an arena $\mathcal{A}$ and Ξ a sequence of addresses. We say that a move $m = \xi \cdot a$ *belong to* Ξ if there exists an address ξ' occurring in Ξ that is a prefix of ξ. The *restriction of a play* s to a sequence of address Ξ is denoted $s_{\restriction \Xi}$, and corresponds to the play s' made of the moves of s that belong to Ξ and such that $p_{s \restriction \Xi}(s_j) = min\{s_i \in proj(s_j) \mid s_i \textit{ belong to } \Xi\}$.

Definition 1.7 (Prefix)**.** Given two strings s and t over an alphabet Σ, s is a *prefix* of t if there exists a string s' over Σ such that $t = s \cdot s'$. In that case we write $s \sqsubseteq t$. Given two legal plays s and t of an arena A we denote $s \sqsubseteq^{\mathsf{O}} t$ if s is an O–ending prefix of t and we denote $s \sqsubseteq^{\mathsf{P}} t$ when s is an P–ending prefix of t. The *longest common prefix* between two strings s and t is denoted $s \wedge t$ and is defined as the longest string u such that $u \sqsubseteq s$ and $u \sqsubseteq t$.

The *last move* or *conclusion* of a play $s = (s_1, \dots, s_n)$ is denoted s_ω and corresponds to the move s_n. The *immediate prefix* of a non-empty string $s = (s_0, \dots, s_n)$ is denoted s^- or $\mathrm{ip}(s)$ and corresponds to $(s_0, \dots, s_{n-1})$. We say that a play s is *justified* if its last move s_ω points to another move s_i belonging to s. Given a justified play $s = (s_1, \dots, s_n)$ and s_i the justification of s_ω the *justification prefix* of s is denoted $\mathrm{jp}(s)$ and corresponds to $(s_1, \dots, s_i)$.

We say that a string s is an *immediate extension* of a string t if s^- correspond to t. Given a string s we denote by $\mathrm{ie}(s)$ its set of immediate extensions. Given a play s if s_ω enables a move m we denote by $s \cdot m$ the play made of the sequence $(s_1, \dots, s_n, m)$ such that m points to s_ω.

Definition 1.8 (State)**.** The *state* of a play s belonging to the interaction $I(A, B, C)$ on the component $A \Rightarrow B$, is denoted $\mathsf{S}_{A \Rightarrow B}(s)$, and corresponds to an actor polarity of $\{\mathsf{P}, \mathsf{O}\}$ it is the actor polarity of the moves m belonging to $A \Rightarrow B$ such that $s \cdot m$ is still in the interaction of A, B and C (indeed this polarity is unique by alternance property).

The *state* of a play s belonging to the interaction $I(A, B, C)$ is denoted $\mathsf{S}_{A,B,C}(s)$ or $\mathsf{S}_{.}(s)$ if there is no ambiguity, and corresponds to the tuple $(\mathsf{S}_{A\Rightarrow B}(s), \mathsf{S}_{B\Rightarrow C}(s), \mathsf{S}_{A\Rightarrow C}(s))$.

1.2 Construction on games

Definition 1.9 (Tensor of games). Given two arena's $A = \langle M_A, \lambda_A, \vdash_A, I_A\rangle$ and $B = \langle M_B, \lambda_B, \vdash_B, I_B\rangle$ and two integers $i \neq j$ their *tensor product* localized on i, j is denoted $A \otimes_{i,j} B$ and defined as follow:

- A move of $A \otimes_{i,j} B$ is either a move of $i \cdot A$ or a move of $j \cdot B$, meaning $M_{A\otimes_{i,j}B} = M_{i\cdot A} \uplus M_{j\cdot B}$.
- $\lambda_{A\otimes_{i,j}B} = [\lambda_{i\cdot A}, \lambda_{j\cdot B}]$.
- A move m enables another move n in $A \otimes_{i,j} B$ if it enables it in the sub–arena $i \cdot A$ or $j \cdot B$. Meaning $\vdash_{A\otimes_{i,j}B}=\vdash_{i\cdot A} \uplus \vdash_{j\cdot B}$.
- The initial moves of $A \otimes_{i,j} B$ are the initial moves of $i \cdot A$ or $j \cdot B$ meaning $I_{A\otimes_{i,j}B} = I_{i\cdot A} \uplus I_{j\cdot B}$.

Definition 1.10 (Implication of games). Given two arena's $A = \langle M_A, \lambda_A, \vdash_A, I_A\rangle$ and $B = \langle M_B, \lambda_B, \vdash_B, I_B\rangle$ and two integers $i \neq j$ their *implication* localized on i, j is denoted $A \multimap_{i,j} B$ and defined as follow:

- A move of $A \multimap_{i,j} B$ is either a move of $i \cdot A$ or a move of $j \cdot B$, meaning $M_{A\multimap_{i,j}B} = M_{i\cdot A} \uplus M_{j\cdot B}$.
- $\lambda_{A\multimap_{i,j}B} = [\overline{\lambda_{i\cdot A}}, \lambda_{j\cdot B}]$.
- A move m enables another move n in $A \multimap_{i,j} B$ if it enables it in the sub–arena $i \cdot A$ or $j \cdot B$, or if m is a initial move of $j \cdot B$ while n is an initial move of $i \cdot A$. Meaning $\vdash_{A\multimap_{i,j}B}=\vdash_{i\cdot A} \uplus \vdash_{j\cdot B} \uplus I_{j\cdot B} \times I_{i\cdot A}$.
- The initial moves of $A \multimap_{i,j} B$ are the initial moves of $j \cdot B$ meaning $I_{A\multimap_{i,j}B} = I_{j\cdot B}$.

Proposition 1.1 (Tensor switching). *Given two arena's A and B and sab a play of $A \otimes_{0,1} B$ If a and b are in different component then b is an opponent move.*

Proof. By induction on the length of s. If $s = \varepsilon$ its state is (O, O), and the opponent must play. In that case the opponent may either play an initial move m in A or in B resulting in the play $\varepsilon \cdot m$ with state (P, O) or (O, P). Without loss of generality assume m is an initial move in A, then the player must respond in A i.e. play a move in A. Indeed, if we assume otherwise that the player plays a move n in B, then $m \cdot n_{\restriction B} = n$ must be a play of B, but n is a player move of B and hence cannot be an initial move, and so $m \cdot n_{\restriction B}$ cannot be a play of B. Thus the player must respond in the arena A.

Now assume s is of size $n + 1$ i.e of the form $s \cdot m$. Assume that m is a player move, if it is a question that is not justified then a must be justified by m otherwise there is two pending question in sma hence, a is enabled by m and no previous question otherwise they'd be two pending question in sm. Therefore a is a move in the arena of m. If a is a question, the move b must be justified for the same reason (otherwise they'd be two pending question), thus b is enabled by a and so b belongs to the arena of the move a. □

Proposition 1.2 (Switching for the implication). *Given A and B two arenas and sab a legal play on $A \Rightarrow B$. If the moves a and b are in different components then b is a player move.*

Proof. The starting configuration of a play is the following, the opponent is to play in the state $(A : \mathsf{P}, B : \mathsf{O})$. Initially then the opponent must play in B thus the state of the play becomes $(A : \mathsf{P}, B : \mathsf{P})$ with player to play. Two things may occur:

- if the player plays in B we return in the same configuration as before, the opponent is to play in the state $(A : \mathsf{P}, B : \mathsf{O})$.
- Otherwise if P plays in A we obtain the state $(A : \mathsf{O}, B : \mathsf{P})$ where opponent is to play, thus the opponent must play in A. Thus we get back to the configuration where player must play and the state is $(A : \mathsf{P}, B : \mathsf{P})$.

This shows that only a player move can change in which component the game is played. □

1.3 Views in a play

Definition 1.11 (Player and opponent view of a play). The player view of a play s is denoted $\lceil s \rceil$ and defined inductively as:

- $\lceil s \rceil = s_\omega$ is s_ω is an initial move.
- $\lceil s \rceil = \lceil \mathrm{jp}(s) \rceil s_\omega$ if s_ω is a non–initial opponent move.
- $\lceil s \rceil = \lceil \mathrm{ip}(s) \rceil s_\omega$ is s_ω is a non–initial player move.

Proposition 1.3 (Views are idempotent). *Given s a pointed string $\lceil\lceil s \rceil\rceil = \lceil s \rceil$ and $\lfloor\lfloor s \rfloor\rfloor = \lfloor s \rfloor$.*

Proof. We do so by induction on the length of s – the base case for an empty play is clear. Assume s is of size n, we treat three possible case for s depending on the type of its last move. Assume that s_ω is an initial move. Then $\lceil s \rceil = s_\omega$ furthermore $\lceil s_\omega \rceil = s_\omega$ since s_ω is the ultimate move of the play s_ω and is by assumption an initial move.

If now s_ω is a non–initial opponent move $\lceil s \rceil = \lceil \mathrm{jp}(s) \rceil s_\omega$ applying again the player view we obtain where the head of $\mathrm{jp}(s)$ is the justifier of s_ω in s. then this is still true in $\lceil s \rceil$ and thus $\mathrm{jp}(\lceil \mathrm{jp}(s) \rceil) = \lceil \mathrm{jp}(s) \rceil$. To conclude call the induction hypothesis to derive $\lceil\lceil \mathrm{jp}(s) \rceil\rceil = \lceil \mathrm{jp}(s) \rceil$.

If s_ω is a non–inital player move. $\lceil s \rceil = \lceil \mathrm{ip}(s) \rceil s_\omega$ and so $\lceil\lceil s \rceil\rceil = \lceil \mathrm{ip}(\lceil s \rceil) \rceil s_\omega$. Now note that $\mathrm{ip}(\lceil s \rceil)$ is exactly $\lceil \mathrm{ip}(s) \rceil$. To conclude call the induction hypothesis to derive $\lceil\lceil \mathrm{ip}(s) \rceil\rceil = \lceil \mathrm{ip}(s) \rceil$. □

Proposition 1.4 (legality of views induce legality). *Given s a legal play of an arena A. For any player move a of A, if $\lceil s \rceil a$ is a legal play over A then so is sa. For any opponent move a of A, if $\lfloor s \rfloor a$ is a legal play over A then so is sa.*

Proof. We treat the case of the player move (the case of the opponent move is similar). Consider s a legal play on A and a a player move of A and assume $\lceil s \rceil a$ is also a legal play. First note that sa is well justified indeed s is well justified and since $\lceil s \rceil a$ and sa have the same pointer on a then sa is also well–justified. Let us show alternance is verified in sa, since s is a legal play it is alternated and so that sa is alternated it must be that the last move of s is an opponent move. By absurdum, assume the last move of s is a player move and denote $s = s'b$ then $\lceil sa \rceil = \lceil s'ba \rceil = \lceil s'b \rceil a = \lceil s' \rceil ba$ hence $\lceil s' \rceil ba$ is also failing alternation. But $\lceil s \rceil a = \lceil s'b \rceil a = \lceil s' \rceil ba$ is supposed to be a legal play and thus to be alternated. We have a contradiction, so we conclude the last move of s is an opponent move making the pointed string sa alternated. The visibility condition is satisfied in $\lceil s \rceil a$ and s by assumption, hence it is also satisfied in sa since the pointer on a in sa and $\lceil s \rceil a$ are related. If a is a question then since s has no answer memory so does sa. If a is an answer it must answer to the pending question in $\lceil s \rceil$ since $\lceil s \rceil a$ is a legal play. Furthermore s and $\lceil s \rceil$ have the same pending question (calling the previous lemma) thus a answers to the pending question of s in sa. This fact together with the assumption that s has no answer memory yields that sa has no answer memory.

We have shown that sa is well–justified, alternated, satisfies visibility and has no answer memory, hence sa is a legal play by definition.

□

Proposition 1.5 (Views conserve legality). *For any legal play s over a game A, $\lfloor s \rfloor$ and $\lceil s \rceil$ are both legal play on A.*

Proof. Consider s a legal game over an arena A. Now let us show that $\lceil s \rceil$ is a also a legal play over A. We proceed by induction on the length of s. The base case is trivial if the length of s is null then $s = \varepsilon$ and $\lceil \varepsilon \rceil = \varepsilon$ hence since ε is a legal game we conclude. For the inductive step consider a legal game of the form sm now we will treat three case based on the nature of the move m.

- If m is a player move then by definition of the player view $\lceil sm \rceil = \lceil s \rceil m$, since by induction $\lceil s \rceil$ is a legal play we must only show that it is legal to extend $\lceil s \rceil$ by m. First, since sm is a legal play and m is a player move $\lceil s \rceil m$ is alternated. For the justification, since sm is a legal play m has a justifier in s and so it has a justifier in $\lceil s \rceil$. Furthermore thanks to the previous lemma s and $\lceil s \rceil$ have the same pending question. This together with the legality of the game sm, implies $\lceil s \rceil m$ still has no answer memory. For the visibility condition, in sm the justifier of m occurs in s in $\lceil s \rceil m$ the justifier of m occurs in $\lceil \lceil s \rceil \rceil$ which corresponds to $\lceil s \rceil$ since views are idempotent.

- Assume now that m is an initial opponent move. Then $\lceil sm \rceil = m$ which is a legal play.
- Assume that m is a non initial opponent move justified by a move n in the pointed string sm. Then sm is of the form $s'ntm$ and $\lceil sm \rceil = \lceil s'ntm \rceil = \lceil s'n \rceil m$. Note that since n justifies m in sm and that sm is legal this ensures that n is a player move, hence $\lceil s'n \rceil m = \lceil s' \rceil nm$. To conclude note that $s'n$ being a substring of sm is a legal play, so by induction $\lceil s' \rceil n$ is a legal play. To conclude we merely note that it is legal to extend $\lceil s' \rceil n$ by m.

□

Definition 1.12 (Threads)**.** A move m is *hereditarily justified* by the occurrence of a move n in a pointed string $(s, \curvearrowright_s)$ if m is related to n in the transitive and reflexive closure of the pointer relation i.e. $m \curvearrowright_s^* n$. A *thread* is a play such that each of its moves are hereditarily justified by the same initial move s_0. Given a play s and n one of its move, the *thread of* n in s is denoted $s_{\upharpoonright n}$ and corresponds to all the moves of s hereditarily justified by n. Given a play s and I a subset of the occurrences of its initial moves, the *thread induced by* I in s corresponds to the pointed string made of the moves in I or hereditarily justified by a move of I.

Proposition 1.6 (Switching)**.** *Given s a legal play of an arena A, I and J two sets which partitions the set of occurrences of initial moves of s. For any moves x and y in $s_{\upharpoonright I}$, such that x justifies y, there exists an integer k such that:*

$$x \dots y_{\upharpoonright J} = \prod_{1 \le i \le k} m_i^- \dots m_i$$

Where for each index i, m_i is justified by the move ${m_i}^-$ and is a player move.

Proposition 1.7 (Commutation of views and initial restriction)**.**

Proposition 1.8 (Initial restriction are legal)**.**

2 Strategies and their composition

2.1 Composition of strategies

Definition 2.1 (Simple plays)**.** Given an arena A a set of plays P_A over A is said to be *simple* if it has the following property:

- non−empty The empty strategy ϵ belongs to P_A.
- prefix−closed If s is a play of P_A then $s' \sqsubseteq s$ belongs to P_A.
- alternating For all $s = (s_1, \dots, s_n)$ of P_A and all index $1 \le i \le n$ we have $pol_A(s_i) \ne pol_A(s_{i+1})$.
- negative For all $s = (s_1, \dots, s_n)$ of P_A the first element is played by the opponent i.e. $pol_A(s_1) = O$.

Definition 2.2 (Strategies)**.** A *simple strategy* σ over a simple game (A, λ_A, P_A) is a set of simple plays $\sigma \subset P_A$ that is:

- non−empty The empty strategy ϵ belongs to σ.
- prefix−closed If s is a play of σ then $s' \sqsubseteq s$ belongs to σ.
- deterministic Given s a play of σ if both sa^+ and sb^+ belong to σ then the moves a^+ and b^+ are equals.
- receptive Given s a play of σ, if sa^- is a simple play then sa^- belongs to σ

An *alternating strategy* over an arena A is a set of legal plays σ over A that is

- NON–EMPTY The empty strategy ϵ belongs to σ.
- PREFIX–CLOSED If s is a play of σ then $s' \sqsubseteq s$ belongs to σ.
- DETERMINISTIC Given s a play of σ if both sa^+ and sb^+ belong to σ then the moves a^+ and b^+ are equals.
- RECEPTIVE Given s a play of σ, if sa^- is a legal play then sa^- belongs to σ.

Note that the only difference between an alternating and a simple strategy is that these strategies consists of subset of different sets of plays, hence the notion of receptiveness also slightly changes from a definition to another. If σ is a strategy over an arena A we write $\sigma : A$.
The *domain* of a strategy σ is the set of plays so where s is a play of σ and so^- is a legal play.

$$dom(\sigma) = \{so^- \mid s \in \sigma \textit{ and } so^- \textit{ is a legal play}\}.$$

Definition 2.3 (Interaction)**.** An *interaction* over the games $\mathcal{A}, \mathcal{B}$ and C is a play u on the set of moves $0 \cdot \mathcal{A} \uplus 1 \cdot \mathcal{B} \uplus 2 \cdot C$ such that $u_{\upharpoonright \mathcal{A},\mathcal{B}}$ is a legal play in $A \Rightarrow B$, $u_{\upharpoonright \mathcal{B},C}$ is a legal play in $B \Rightarrow C$ and $u_{\upharpoonright \mathcal{A},C}$ is a legal play in $A \Rightarrow C$. We denote by $I(\mathcal{A}, \mathcal{B}, C)$ the set of interactions on $\mathcal{A}, \mathcal{B}$ and C. Given u a word with pointers belonging to $I(A, B, C)$ $u_{\upharpoonright A,B}$ and $u_{\upharpoonright B,C}$ are its *internal projections* while $u_{\upharpoonright A,C}$ is its *external projection.*

Given two strategies $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ their *composite* or *interaction* is denoted $\sigma \parallel \tau$ and is defined as:

$$\sigma \parallel \tau = \{u \in I(A, B, C) \mid u_{\upharpoonright A,B} \in \sigma^{0,1} \textit{ and } u_{\upharpoonright B,C} \in \tau^{1,2}\}$$

The *composition* of two strategies $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ is denoted $\tau \odot \sigma$ and is defined as:

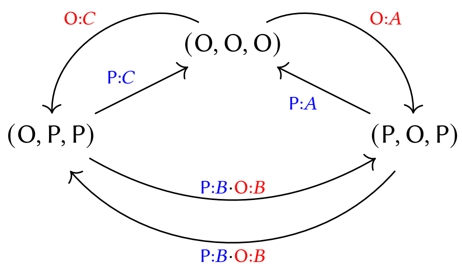
$$(\tau \odot \sigma)^{0,2} = \{u_{\upharpoonright A,C} \mid u \in \sigma \parallel \tau\}$$

Hence it is the set of strategies belonging to the interaction of σ and τ restricted to the address of A and C. A play u is a *witness* of another play s whenever there exists some addresses Ξ such that s corresponds to the restriction of u to the addresses of Ξ, meaning $s = u_{\upharpoonright \Xi}$.

Lemma 2.1 (Unique witness)**.** *Given $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ two strategies, for all play $s \in \tau \odot \sigma$ there exists a unique witness u of $I(A, B, C)$ such that $s = u_{\upharpoonright A,C}$, $u_{\upharpoonright A,B} \in \sigma$ and $u_{\upharpoonright B,C} \in \tau$.*

Proof. Consider u and v two *distinct* witnesses of a play s of the composition $\tau \odot \sigma$. Since $s = u_{\upharpoonright A,C}$ and $s = v_{\upharpoonright A,C}$, u and v can only differ on moves from the arena B. Consider then $t = u \wedge v$ the intersection of the two witnesses, Consider $(u \wedge v)_{\upharpoonright A,B} = u_{\upharpoonright A,B} \wedge v_{\upharpoonright A,B}$ assume it ends by an opponent move this means u and v differ on some B–moves that made the play u or v ending by a player move. □

Proposition 2.1 (state diagram of interactions)**.** *Given three games A, B, C the state diagram of a pointed string in $I(A, B, C)$ is the following:*



Proof. Consider s a pointed string of $I(A, B, C)$. At the beginning of the play, the opponent is to play in the state $(A \Rightarrow B : \mathsf{O}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{O})$. In fact the opponent may only start in C:

- The opponent cannot play a move in A since moves in A are not initial in any of the projection.
- Also the opponent cannot play in B since otherwise for the projection on $B \to C$ the play is not legal (since the move is not justified and not initial because it is a player move in this projection).
- If the opponent plays a move in C then from the point of view of the game $B \Rightarrow C$ and $A \Rightarrow C$ the player must now play.

Hence in that case player is to play in $(A \Rightarrow B : \mathsf{O}, B \Rightarrow C : \mathsf{P}, A \Rightarrow C : \mathsf{P})$, the player can choose to:

- respond in C in which case now in both games $B \Rightarrow C$ and $A \Rightarrow C$ the opponent must play hence the state is now again $(A \Rightarrow B : \mathsf{O}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{O})$.
- respond in A is not possible since, otherwise in the projection $A \Rightarrow B$ the play is not legal (the move is not justified but not initial)
- respond in B, with a move that is justified in the projection $B \Rightarrow C$ and initial in $A \Rightarrow B$. In that case the player must play in $A \Rightarrow B$ and in $A \Rightarrow C$ (since it hasn't changed), on the other hand now $B \Rightarrow C$ the player must play hence the state is now $(A \Rightarrow B : \mathsf{P}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{P})$.
- In that state the opponent is able to play in none of the arenas A, B or C as it would go against legality of the projections.

Thus the opponent must now play in the state $(A \Rightarrow B : \mathsf{P}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{P})$. In the state $(A \Rightarrow B : \mathsf{P}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{P})$. observe that:

- If the player plays in A, then the opponent must play in $(A \Rightarrow B : \mathsf{O}, B \Rightarrow C : \mathsf{O}, A \Rightarrow C : \mathsf{O})$. going back to the initial state.
- The player may not play in C or B otherwise the projection on $B \to C$ is no longer legal.
- If the opponent play in B, now the state in the projections $A \Rightarrow B$ and $B \to C$ get inverted. Thus the player is to play in the state $(A \Rightarrow B : \mathsf{O}, B \Rightarrow C : \mathsf{P}, A \Rightarrow C : \mathsf{P})$.
- The opponent may not play in A or C since player is to play in these projection.

□

Proposition 2.2 (Composition is well–defined). *Given $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ two strategies, then their composition $\tau \odot \sigma$ is also a strategy.*

Proof. We must show that $\tau \odot \sigma$ satisfies the four conditions defining strategies:

non empty: By definition σ and τ are non empty and contain ε. Furthermore, $\varepsilon_{\upharpoonright A,C}$ corresponds to ε. Finally ε is contained in the interface $\sigma \parallel \tau$, hence we conclude that ε is contained in $\tau \odot \sigma$.

prefix–closed: Consider u a play of the composition $\tau \odot \sigma$ and $s \sqsubseteq u$ one of its prefix, let us show s is still in the composition. By definition there exists $u' \in \sigma \parallel \tau$ such that $u'_{\upharpoonright A,C} = u$, since $s \sqsubseteq u$ then there must be $s' \sqsubseteq u'$ such that $s = s'_{\upharpoonright A,C}$. Our goal is then to exhibit such an s' belonging to $\sigma \parallel \tau$. Note that $u'_{\upharpoonright A,B}$ is in σ and $u'_{\upharpoonright B,C}$ is in τ now since s' is a prefix of u' and that the prefix relation is preserved by restriction, we can conclude $s' \in \sigma \parallel \tau$ (because both strategies are prefix closed). Hence since $s = s'_{\upharpoonright A,C}$ the play s is an element of the composition.

deterministic: Consider s a play of the composition $\tau \odot \sigma$ ending by an opponent move, denote $s' \in \sigma \parallel \tau$ its witness i.e. such that $s'_{\upharpoonright A,C} = s$. Consider and sa^+ and sb^+ two plays still belonging to the composition where a^+ and b^+ are two player moves of $A \Rightarrow C$. Let us show that these two plays are the same. Note that $s_1 := s'a^+$ and $s_2 := s'b^+$ are witnesses of sa^+ and sb^+, since $s'a^+_{\upharpoonright A,C} = s'_{\upharpoonright A,C}a^+ = sa^+$, and the same thing goes for sb^+.
If a^+ and b^+ both belong to A, then $s_1 \upharpoonright A, B = s' \upharpoonright A, Ba^+$ and $s_2 \upharpoonright A, B = s' \upharpoonright A, Bb^+$ both belong to σ. Then since σ is deterministic this implies a^+ and b^+ to be equal

(and have the same pointer). In the same way we can conclude if a and b both belong to C using the hypothesis on τ.

If on the other hand a and b belong respectively to A and C let us see what follows. Note that $s_1 = s'a^+$ and $s_2 = s'b^+$. s' is a pointed string of the interaction $I(A, B, C)$ hence thanks to the state diagram lemma, s' is in one of the three state $(\mathrm{O}, \mathrm{O}, \mathrm{O})$, $(\mathrm{P}, \mathrm{O}, \mathrm{P})$ or $(\mathrm{O}, \mathrm{P}, \mathrm{P})$. Hence if both $s_1 = s'a^+$ and $s_2 = s'b^+$ are still in the interaction s' must be in the state $(\mathrm{P}, \mathrm{O}, \mathrm{P})$ but in that case the player may not play in C (because opponent is to play on the projection on $B \Rightarrow C$). which contradicts the fact that $s'b^+$ belongs to the interaction $I(A, B, C)$.

RECEPTIVE Consider s a play of the composition and a^- an opponent move on $A \Rightarrow C$ let us show sa^- is still in the composition, assuming it is a legal play on $A \to C$. By definition s must have a witness s' belonging to $\sigma \parallel \tau$. The move a^- is either a move of A or C, without loss of generality assume it is a move of A. and consider the play $s'a^-$. Note that $s'a^-_{\restriction B,C} = s'_{\restriction B,C}$ and thus belong to τ. On the other hand $s'a^-_{\restriction A,B} = s'_{\restriction A,B}a^-$ now since $s'_{\restriction A,B}$ is in σ, that σ is receptive and a^- is a move of A (and thus of $A \Rightarrow B$), we conclude that $s'a^-_{\restriction A,B}$ is in σ. This show $s'a^-$ belong to $\sigma \parallel \tau$. Finally notice that $s'a^-_{\restriction A,C} = s'_{\restriction A,C}a^- = sa^-$, so we conclude that sa^- has a witness and thus belong to the composition $\tau \odot \sigma$.

□

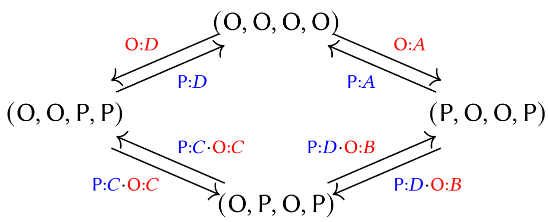
2.2 Composition of strategies is associative

Definition 2.4 (Interaction of arenas)**.** Given a list of arenas $A_1, \ldots, A_n$ their *interaction* is denoted $I(A_1, \ldots, A_n)$ and corresponds to the set of pointed string s over $\bigcup_{1 \le i \le n} M_i$ such that:

INTERNAL LEGALITY For each i the projection $s \restriction A_i \Rightarrow A_{i+1}$ is a legal play on the arena $A_i \Rightarrow A_{i+1}$.

EXTERNAL LEGALITY The projection $s \restriction A_1 \Rightarrow A_n$ is a legal play in the arena $A_1 \Rightarrow A_n$.

Proposition 2.3 (State diagram of interaction of size 4)**.** *Given A, B, C, D four arenas the state diagram of their interaction $I(A, B, C, D)$ is the following.*



Proof. Consider s a pointed string of the interaction $I(A, B, C, D)$. Assume the state of s is in the state $(\mathrm{O}, \mathrm{O}, \mathrm{O}, \mathrm{O})$ then only the opponent may play, the opponent has the following choice:

$\mathrm{O} : A = \checkmark$ The opponent can play in A alternating the polarity in the states $A \Rightarrow B$ and $A \Rightarrow D$ thus entering in the state $(\mathrm{P}, \mathrm{O}, \mathrm{O}, \mathrm{P})$.

$\mathrm{O} : B, C = \times$ The opponent cannot play in the internal components such as B or C because the position of B (resp. C) are opposites in the projections $A \Rightarrow B$ and $B \Rightarrow C$ (resp. in $B \Rightarrow C$ and $C \Rightarrow D$) while the polarity on all the projection is the same.

$\mathrm{O} : D = \checkmark$ The opponent can play in D and thus in the projection $A \Rightarrow D$ and $C \Rightarrow D$ the polarity alternates thus the state becomes $(\mathrm{O}, \mathrm{O}, \mathrm{P}, \mathrm{P})$.

If s is now in the state $(\mathrm{O}, \mathrm{O}, \mathrm{P}, \mathrm{P})$ let us observe what cannot happen.

$\mathrm{P}, \mathrm{O} : A = \times$ The opponent and the player may not play in A since the polarities in $A \Rightarrow B$ and $A \Rightarrow D$ are the same while A has the same position in both projections.

$\mathsf{P},\mathsf{O} : B = \times$ The opponent and the player may not play in B since the polarity on B is the same in the projection $A \Rightarrow B$ and $B \Rightarrow C$ while the position of B is also inverted in both.

$\mathsf{P},\mathsf{O} : C = \checkmark$ The opponent and the player can play in C since the polarity on C alternates the projection $C \Rightarrow D$ and $B \Rightarrow C$ while the position of C is also alternates between the two. In that case the game gets to the state $(\mathsf{O}, \mathsf{P}, \mathsf{O}, \mathsf{P})$.

$\mathsf{O} : D = \times$ The opponent and the player may not play in D since in $A \Rightarrow D$ and $C \Rightarrow D$ the player is to play and D has the same position in both projections.

$\mathsf{P} : D = \checkmark$ The player may play in D as the position of D in $C \Rightarrow D$ and $A \Rightarrow D$ are the same and the player is to play in both. In that case the polarity is inverted in these arenas and the state becomes $(\mathsf{O}, \mathsf{O}, \mathsf{O}, \mathsf{O})$.

If s is in the state $(\mathsf{O}, \mathsf{P}, \mathsf{O}, \mathsf{P})$ let us see what can occur:

$\mathsf{P},\mathsf{O} : A = \times$ The player and the opponent may not play in A because opponent is to play in $A \Rightarrow B$ and player is to play in $A \Rightarrow D$ whiel A as the same position in both arenas.

$\mathsf{P},\mathsf{O} : B = \checkmark$ The player or the opponent may play in B because opponent must play in $A \Rightarrow B$ and player must play in $B \Rightarrow C$, while the position of B is inverted between the two arenas. then the state of the game becomes $(\mathsf{P}, \mathsf{O}, \mathsf{O}, \mathsf{P})$.

$\mathsf{P},\mathsf{O} : C = \checkmark$ The player and the opponent may play in C, since player must play in $B \Rightarrow C$ and opponent must play in $C \Rightarrow D$ and the position of C alternates from an arena to the other. Then the state of the game becomes $(\mathsf{O}, \mathsf{O}, \mathsf{P}, \mathsf{P})$.

$\mathsf{P},\mathsf{O} : D = \times$ The player and the opponent cannot play in D because, player is to play in $A \Rightarrow D$ while opponent is to play in $C \Rightarrow D$ but D has the same position in both projections.

If s is in the state $(\mathsf{P}, \mathsf{O}, \mathsf{O}, \mathsf{P})$ let us see what can occur:

$\mathsf{P} : A = \checkmark$ The player can play in A as player must player in $A \Rightarrow B$ and $A \Rightarrow D$. The state of the game then becomes $(\mathsf{O}, \mathsf{O}, \mathsf{O}, \mathsf{O})$.

$\mathsf{O} : A = \times$ The opponent can not play in A as player must play in both in $A \Rightarrow B$ and $A \Rightarrow D$.

$\mathsf{P},\mathsf{O} : B = \checkmark$ The player and the opponent can play in B as player must player in $A \Rightarrow B$ and opponent must $B \Rightarrow C$. The state of the game then becomes $(\mathsf{O}, \mathsf{P}, \mathsf{O}, \mathsf{P})$.

$\mathsf{P},\mathsf{O} : C = \times$ The player and the opponent can not play in C because opponent must play in both $B \Rightarrow C$ and $C \Rightarrow D$ while C has inverted position in the two projections.

$\mathsf{P},\mathsf{O} : D = \times$ Since the polarity in the projections $A \Rightarrow D$ and $C \Rightarrow D$ are opposites while D has the same position in both projections, no one can play in D.

□

Proposition 2.4 (Zipping)**.** *Given u and v two pointed strings respectively in $I(A, C, D)$ and $I(A, B, C)$ such that $u_{\restriction A,C} = v_{\restriction A,C}$. Then there exists a unique w in $I(A, B, C, D)$ such that $w_{\restriction A,C,D} = u$ and $w_{\restriction A,B,C} = v$.*

Similarly, if u and v are respectively in $I(A, B, D)$ and $I(B, C, D)$ such that $u_{\restriction B,D} = v_{\restriction B,D}$. Then there exists a unique w in $I(A, B, C, D)$ such that $w_{\restriction A,B,D} = u$ and $w_{\restriction B,C,D} = v$.

Proof. We do so by induction on the size of $u_{\restriction A,C}$. If $u_{\restriction A,C} = \epsilon$, then u may only do moves in D (using the state diagram). then we set $u = w$ and conclude.

Otherwise denote by a the final move of $u_{\restriction A,C}$, we apply the induction hypothesis to $u' = u_{<a}$ and $v' = v_{<a}$, yielding a unique witness w' in $I(A, B, C, D)$ such that $w'_{\restriction A,C,D} = u'$ and $w'_{\restriction A,B,C} = v'$.

We want to show that the last move of $u_{\restriction A,C}$ is the last move of at least one of the string u or v. We distinguish two cases:

- If a is a player move in $A \Rightarrow C$ then u may not continue with moves in A or C as a is the last move of $u_{\restriction A,C} = v_{\restriction A,C}$, u may not play in B either because it is in the interaction $I(A, C, D)$. Thus u may then follow only with D moves, so is of the form $u = u_{\leq a} t_u$. Furthermore since a is a player move in $A \Rightarrow C$ hence it is a player move in A or C since v is in $I(A, B, C)$ by looking at the state diagram we arrive in the state $(\mathsf{O}, \mathsf{O}, \mathsf{O})$ in which v may only follow with moves in A or C, but that is not possible since a is the last move of $v_{\restriction A,C}$. Hence a is the last move of v.

- On the other hand if a is an opponent move in $u_{\upharpoonright A,C} = v_{\upharpoonright A,C}$, v (and u) may not follow with moves in A or C, but also v may not follow with moves in D since it belongs to $I(A, B, C)$. Thus v may only follow with B moves, and so is of the form $v = v_{\leq a} t_v$.
  Note that u is in the interaction $I(A, C, D)$ by looking at the state diagram of the interaction $I(A, C, D)$, since a is an opponent move in $A \Rightarrow C$ and thus an opponent move in A or C, in both case u is then in the state (P, O, P) from which u cannot play moves in D. This shows that a is the last move of u.

Now let us observe that in both cases we can construct the witness w in the interaction $I(A, B, C, D)$

- Assume that $u = u_{\leq a} t_u$ and $v = v_{\leq a}$. Consider $w = w' a t_u$ and show it is an accurate witness. Because t_u is made of moves only in D and a is a move in A or C we have $w_{\upharpoonright A,C,D} = w'_{\upharpoonright A,C,D} a t_u$. Then by hypothesis on w' we have $w'_{\upharpoonright A,C,D} = u_{<a}$ thus $w_{\upharpoonright A,C,D} = u_{<a} a t_u = u$. On the other hand $w_{\upharpoonright A,B,C} = w'_{\upharpoonright A,B,C} a = v_{<a} a = v$ since t_u consists of only moves in B it does not appear in this projection.
- Assume that $v = v_{\leq a} t_v$ and $u = u_{\leq a}$. Consider $w = w' a t_v$ and show it is an accurate witness. Because t_v is made of moves only in B and a is a move in A or C we have $w_{\upharpoonright A,B,C} = w'_{\upharpoonright A,B,C} a t_v$. Then by hypothesis on w' we have $w'_{\upharpoonright A,B,C} = v_{<a}$ thus $w_{\upharpoonright A,B,C} = v_{<a} a t_v = v$. On the other hand $w_{\upharpoonright A,C,D} = w'_{\upharpoonright A,C,D} a = u_{<a} a = u$ since t_v consists of only moves in B and so does not appear in this projection.

For the symmetrical result we perform the same proof "up to renaming". □

Proposition 2.5 (Composition is associative)**.** *Given three strategies $\sigma : A \Rightarrow B$, $\tau : B \Rightarrow C$ and $\nu : C \Rightarrow D$, the strategies $(\nu \odot \tau) \odot \sigma$ and $\nu \odot (\tau \odot \sigma)$ are the same.*

Proof. Consider s a play of $\nu \odot (\tau \odot \sigma)$ by definition s has a witness u in $\nu \parallel (\tau \odot \sigma)$ i.e. $u_{\upharpoonright A,D} = s$. Furthermore it means that u is in the interaction $I(A, C, D)$ with $u_{\upharpoonright A,C}$ belong to $\tau \odot \sigma$ while $u_{\upharpoonright C,D}$ is in ν. By the definition of the composition there exists a witness v in $\tau \parallel \sigma$ such that $v_{\upharpoonright A,C} = u_{\upharpoonright A,C}$.

By the zipping lemma there exists a unique w in $I(A, B, C, D)$ such that $w_{\upharpoonright A,C,D} = u$ and $w_{\upharpoonright A,B,C} = v$. Note that $w \upharpoonright A, B = w \upharpoonright A, B, C \upharpoonright A, B = v_{\upharpoonright A,B}$ and so is in σ. Also $w \upharpoonright B, C = w \upharpoonright A, B, C \upharpoonright B, C = v_{\upharpoonright B,C}$ and so is in τ. Finally $w \upharpoonright C, D = w \upharpoonright A, C, D \upharpoonright C, D = u_{\upharpoonright C,D}$ is in ν.

We can show that $w_{\upharpoonright B,D}$ is alternating. By the state diagram of the interaction $I(A, B, C, D)$ we know that in between any B–moves and D–moves there must be one move from A or C. Now since $w_{\upharpoonright A,B,C}$ is a legal play after a sequence of B moves in w followed by a sequence of D moves. In between the two sequences since $w_{\upharpoonright A,B,C}$ is a legal play after there must be an alternating sequence of move in A or C but looking at the state diagram again this sequence is made of moves only in A or C (as moves in B and D cannot be made) of uneven length.

On the other hand since $w_{\upharpoonright A,C,D}$ is a legal play after the sequence of moves in A and C the move in D must first respond to m.

Thus w is in $\nu \parallel \tau$, and so by definition $w \upharpoonright B, C, D \upharpoonright B, D = w_{\upharpoonright B,D} \in \nu \odot \tau$. Furthermore, since $w_{\upharpoonright A,B}$ is in σ and $w_{\upharpoonright A,D}$ is a legal play (since $w \in I(A, B, C, D)$), we conclude that $w_{\upharpoonright A,D}$ belong to $(\nu \odot \tau) \odot \sigma$. But note that $w \upharpoonright A, D = w \upharpoonright A, C, D \upharpoonright A, D = u \upharpoonright A, D = s$ thus we can finally conclude that s belongs to $(\nu \odot \tau) \odot \sigma$.

For the inclusion of $(\nu \odot \tau) \odot \sigma$ in $\nu \odot (\tau \odot \sigma)$ the proof is symmetric.

□

2.3 The copycat strategy

Definition 2.5 (Identity, the copycat strategy)**.** The *identity* or *copycat* of an arena A, is denoted id_A and is the strategy defined as:

$$id_A = \{s \in \mathcal{L}_{0 \cdot A \Rightarrow 1 \cdot A} \mid \textit{For any player–prefix } s' \textit{ of } s, s'_{\upharpoonright 0} = s'_{\upharpoonright 1}\}$$

Lemma 2.2 (The copycat is neutral with respect to composition). *Given a strategy $\sigma : A \Rightarrow B$ we have that $\sigma \odot id_A = \sigma$ while $id_B \odot \sigma = \sigma$.*

Proof. **1** $\Rightarrow$ **2**. Consider s a play belonging to the strategy $id_A \odot \sigma$ by definition this means there exists a witness u of s belonging to the interaction $id_A \parallel \sigma$, meaning $u_{\upharpoonright 0,B} = s$. Since u belong to the interaction this means $u_{\upharpoonright 0,1}$ is a play of id_A while $u_{\upharpoonright 1,2}$ is a play of the strategy σ. Our method to conclude that s is in the strategy σ is to show that $s = u_{\upharpoonright 1,2}$.

Since $u_{\upharpoonright 0,1}$ belong to id_A by definition we have $u \upharpoonright 0,1 \upharpoonright 0 = u \upharpoonright 0,1 \upharpoonright 1$. Calling the restriction lemma we conclude that $u_{\upharpoonright 0} = u_{\upharpoonright 1}$. From that we then derive $u_{\upharpoonright 0,2} = u_{\upharpoonright 1,2}$, and thus $s = u_{\upharpoonright 1,2}$.

1 $\Rightarrow$ **2**. On the other hand if s is a strategy of $\sigma : A \Rightarrow B$, we proceed by induction on the length of the play s. If s is made of one move m belonging to the arena $A \Rightarrow B$, since s is a legal play m must be an initial move of $1 \cdot A \to 2 \cdot B$ hence m is an initial move of $M_{2\cdot B}$. Let us show that m is in the interface of $0 \cdot A$, $1 \cdot A$ and $2 \cdot B$:

- First, the restriction $m_{\upharpoonright 0\cdot A, 1\cdot A}$ is the empty sequence and hence is a legal play of the implication $0 \cdot A \to 1 \cdot A$.
- Secondly, since m is an initial move in the arena B we have $m_{\upharpoonright 0\cdot A,2\cdot B} = m_{\upharpoonright 1\cdot A,2\cdot B} = m$. And again, since m is an initial move of $1 \cdot A \to 2 \cdot B$ and $0 \cdot A \to 2 \cdot B$ it is a legal play of these two arena's.

Now let us show that m belong to the interaction of the strategies σ and id_A (i.e. that the respective internal restrictions belong to the the respective strategies).

- First, note that $m_{\upharpoonright 1,2} = m$ is made of only one opponent move, it is thus a legal play, since any strategy is receptive and contains the empty play it must contain $\varepsilon \cdot m$ which is m. In particular m belongs to σ.
- Secondly, $m_{\upharpoonright 0,1}$ is the empty sequence and hence is a play of id_A, as obviously $\varepsilon_{\upharpoonright 0} = \varepsilon_{\upharpoonright 1} = \varepsilon$.

Finally let us find a witness of m in this interface, well indeed $m_{\upharpoonright 0,2} = m$ and hence provides a witness of m meaning that m is a play of $id_A \odot \sigma$.

Now let us proceed with the induction assuming $s = (s_1, \dots, s_n, a, b)$ is a play of $\sigma : A \Rightarrow B$. Since σ is closed by prefix then $s' = (s_1, \dots, s_n)$ is a play of σ and so calling the induction hypothesis, it is a play of the composition $id_A \odot \sigma$ meaning it has a witness u' in $id_A \parallel \sigma$.

Note that the moves a and b may be moves either of $1 \cdot A$ or $2 \cdot B$. We will treat all the cases possible, and show that in each case we can construct u a witness of s in $id_A \parallel \sigma$.

- If a and b both belong to $1 \cdot A$ then consider the play $u = u' \cdot (1 \cdot a, 0 \cdot a, 0 \cdot b, 1 \cdot b)$. Note that $u_{\upharpoonright 0,1}$ corresponds to $u'_{\upharpoonright 0,1}aabb$ where $u'_{\upharpoonright 0,1}$ is a play of the copycat on $0 \cdot A \Rightarrow 1 \cdot A$, and thus it follows that $u'_{\upharpoonright 0,1}aabb$ is also in the copycat strategy. This means $u_{\upharpoonright 0,1} \in id_A$.
  Now lets show u is also in $\sigma : 1 \cdot A \Rightarrow 2 \cdot B$. Indeed note that $u_{\upharpoonright 1,2} = u'_{\upharpoonright 1,2}ab$, and that since u' is a witness $u'_{\upharpoonright 1,2} = s'$ thus $u_{\upharpoonright 1,2} = s'ab$ which by assumption is in σ.
  Furthermore $u_{\upharpoonright 0,2} = s'ab$ and thus is a legal play, u is a witness of s in $id_A \parallel \sigma$ meaning that $s \in \sigma \odot id_A$.
- If a and b both belong to $2 \cdot B$ then consider the play $u = u'ab$. Then $u_{\upharpoonright 0,1} = u'_{\upharpoonright 0,1}$ is in the copycat strategy. On the other hand $u_{\upharpoonright 1,2} = u'_{\upharpoonright 1,2}ab$ and thus corresponds to $s'ab$ which by assumption is in σ. Furthermore $u_{\upharpoonright 0,2} = s'ab$ and thus is a legal play on $0 \cdot A \Rightarrow 2 \cdot B$. Hence u is a witness of s in $id_A \parallel \sigma$, meaning that $s \in \sigma \odot id_A$.
- If a and b belong to two distinct components – say respectively $1 \cdot A$ and $2 \cdot B$. Note that due to the switching property for implication a is in the same component as s_n. now consider $u = u' \cdot (1 \cdot a, 0 \cdot a, 0 \cdot b, 2 \cdot b)$.
  Now note that $u_{\upharpoonright 0,1} = u'_{\upharpoonright 0,1}aa$ and is indeed in the copycat strategy on A. On the other hand $u_{\upharpoonright 0,1} = u'_{\upharpoonright 1,2}ab$ which corresponds to $s'ab$ by assumption this is an element of σ. Furthermore $u_{\upharpoonright 0,2} = u'_{\upharpoonright 0,2}ab$ which also corresponds to $s'ab = s$ and is therefore a legal play. This shows that s has a witness in $id_A \parallel \sigma$ and so it means $s \in \sigma \odot id_A$.

Since in each case we were able to construct a witness we conclude $s \in s \odot id_A$. For showing that $\sigma = id_B \odot \sigma$ we follow the same argument. □

3 Enriching Strategies: visibility, well bracketing and innocence

3.1 Wining strategies

Definition 3.1 (Winning strategy)**.** A strategy $\sigma : A \Rightarrow B$ is wining if for any $s \in \sigma$ and some opponent move a such that sa is a legal play then there exists b a player move such that sab is still in σ.

Proposition 3.1 (Composition of winning strategies)**.** *Given two wining strategies $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$. Their composition $\tau \odot \sigma$ is also a wining strategy.*

Proof. Consider s some play belonging to $\tau \odot \sigma$, and a an opponent move of $A \Rightarrow C$ such that sa is a legal play. Let's show the existence of b such that sab is in $\tau \odot \sigma$.

By definition s has a witness u in $\tau \parallel \sigma$. This means in particular that $u_{\upharpoonright A,B}$ is in σ while $u_{\upharpoonright B,C}$ is in τ. Now the opponent move a may either be in A or C.

If $a = m_0$ in is A then $ua_{\upharpoonright A,B} = u_{\upharpoonright A,B}a$ which is a legal play, thus since σ is winning there must be a player move m_1 in $A \Rightarrow B$ such that $u_{\upharpoonright A,B}m_0m_1$ is in σ.

- If m_1 is in the same component as m_0 (in this case the external component A) then $um_0m_1 \upharpoonright B, C = u_{\upharpoonright B,C}$ and is in τ. In that case um_0m_1 belong to $\sigma \parallel \tau$ and $um_0m_1 \upharpoonright A, C = sm_0m_1 = sab$ thus its a witness of sab. This shows $sab \in \tau \odot \sigma$ and allows us to conclude.
- If m_1 is in the internal component B while m_0 is in A. Then note that $um_0m_1 \upharpoonright B, C = u_{\upharpoonright B,C}m_1$. Let us point out that m_1 is a player move in $A \Rightarrow B$ and thus it is a opponent move in $B \Rightarrow C$. Since $u_{\upharpoonright B,C}$ is in the winning strategy τ then there exists m_2 in $B \Rightarrow C$. such that $u_{\upharpoonright B,C}m_1m_2$ belongs to τ. In particular $um_0m_1m_2 \upharpoonright B, C \in \tau$. Now we have to test is $um_0m_1m_2 \upharpoonright A, C$ is in σ. This again depends on whether m_2 is in an internal or external component.

We then call again the same argument on m_2 and thus recursively we can create a sequence of moves $(m_i)_{i \in \mathbb{N}}$ such that $um_0m_1 \dots m_n \upharpoonright A, C$ is a legal play. If one of the move m_n is in an external component A or C we can conclude.

Otherwise there are infinite plays $v_1 = um_0 \cdot m_1 \cdot \dots_{\upharpoonright A,B} \in \sigma$ and $v_2 = um_0 \cdot m_1 \cdot \dots_{\upharpoonright B,C} \in \tau$ while $u_{\upharpoonright A}$ and $u_{\upharpoonright C}$ are finite. And furthermore $v_1 \upharpoonright B = v_2 \upharpoonright B^{\perp}$. Player must then lose in one of the play v_1 or v_2 contradicting the fact that the strategies are winning. Therefore we conclude that there exists necessarily a witness of $sab \in \tau \odot \sigma$.

□

3.2 History-free Strategy

Definition 3.2 (History free strategy)**.** A strategy $\sigma : A \Rightarrow B$ is history free if whenever sab and ta are two plays of σ where a is an opponent move, then tab is in σ.

Proposition 3.2 (Composition of history free strategy)**.** *Given $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ two history free strategies. Their composition $\tau \odot \sigma$ is still a history free strategy.*

Proof. Consider two strategies sac and ta belonging to $\tau \odot \sigma$ let $ua\bar{b}c$ and va be their respective witness. Let's show that tac also have a witness. a is an opponent move in A or C we assume without loss of generality that it belongs to A.

If c is also in A we have $vab_{\upharpoonright A,B} = v_{\upharpoonright A,B}ac \in \sigma$ and $vab_{\upharpoonright B,C} = v_{\upharpoonright B,C} \in \tau$ □

$$\begin{tikzcd}
cIA & cPCF \arrow[l, "+interference"'] & Vis \arrow[r, "+innocence"] \arrow[d, "+well-bracketing"'] & Vis(Inn) \arrow[d, "+well-bracketing"] \\
IA \arrow[u, "+control"] & PCF \arrow[l, "+interference"] \arrow[u, "+control"'] & Vis(WB) \arrow[r, "+innocence"'] & Vis(Inn, WB)
\end{tikzcd}$$

Figure 1: The semantic cube taken from Pierre Clairambault "habilitation" [Cla22]. Adding innocence from the strategies corresponds to removing interference in the programming language, on the other hand, adding well–bracketing to the strategies corresponds to removing control in the programming language.

3.3 Visible strategies and their composition

Definition 3.3 (Visibility)**.** A legal play s is P–*visible* (resp. O–*visible*) if its player view $\lceil s \rceil$ (resp. opponent view $\lfloor s \rfloor$) is still a legal play. A strategy $\sigma : A$ is P–*visible* (resp. O–*visible*) if all its plays are P–visible (resp. O–visible).

Lemma 3.1 (Visibility and internal components)**.** *Given an interaction $I(A, B, C)$ of three arena's A, B and C. Let u be an element of $I(A, B, C)$, and X a component of the last move of u (either $A \Rightarrow B$ or $B \Rightarrow C$). If for each prefix $u' \sqsubseteq u$ the last move of u' points in $\lceil u' \rceil$, then $\lceil u_{\restriction X} \rceil$ is contained in $\lceil u \rceil$.*

Proof. We do so by induction on u. If u is of size 1 then its first move is an initial move in $A \Rightarrow C$ and thus $\lceil u \rceil = u$. Furthermore $u_{\restriction X}$ is either the empty sequence or u itself granting the inclusion.

Now if we assume u to be of size $n + 1$ we treat cases depending on the type of the last move:

- If the last move of u is initial on X. then $u_{\restriction X}$ contains the last move of u as its last move, and since the move is initial $\lceil u \rceil = \lceil u_{\restriction X} \rceil$.
- If the last move is a player move on X then u is of the form $u'a$ and $\lceil u'a_{\restriction X} \rceil = \lceil u'_{\restriction X} \rceil a$. Note that the last move of u' is still a move in X: assuming that X is $A \Rightarrow B$ if the player just played in X then the previous opponent move was either a move in A or B. Thus we can call the induction hypothesis on u' and claim that $\lceil u'_{\restriction X} \rceil$ is included in $\lceil u' \rceil$. Hence $\lceil u'_{\restriction X} \rceil a$ is contained in $\lceil u' \rceil a = \lceil u \rceil$.
- If the last move is an opponent move on X that is not initial. Then u is of the form $u_1 b u_2 a$ where a points on b. By hypothesis on u the move b is in the player view $\lceil u \rceil$. Also $\lceil (u_1 b u_2 a)_{\restriction X} \rceil = \lceil u_1 b_{\restriction X} \rceil a$, and by induction hypothesis (since $u_1 b$ is a substring of u) $\lceil u_1 b_{\restriction X} \rceil$ is a subsequence of $\lceil u_1 b \rceil$. Thus $\lceil u_1 b_{\restriction X} \rceil a$ is a subsequence of $\lceil u_1 b \rceil a$ which is $\lceil u \rceil$ and so we conclude.

□

Lemma 3.2 (Visibility in interactions)**.** *Given three arena's A, B and C and a play belonging to $I(A, B, C)$. If $u_{\restriction A,B}$ and $u_{\restriction B,C}$ are both* P*–visible, then:*

1. *For any prefix ta of u the move a points in $\lceil ta \rceil$.*
2. *the player view $\lceil u \rceil$ is also in the interaction $I(A, B, C)$*

Proof. We reason by induction. If u is made of only one move then its only move is initial. Thus the proposition is trivial as the only prefix of u is itself or ϵ. Now consider $u = va$ of size $n + 1$ and treat the three possible cases depending on the nature of the last move of u:

- If the last move of $u = va$ is an initial opponent move then a prefix of u is either a prefix of v in which case we conclude by induction. Otherwise it is va itself but all elements in v verify the condition whereas a has no pointer thus the assumption is true. Furthermore $\lceil va \rceil = a$ and is the interaction.

- If the last move of u is a non initial external opponent move, meaning an opponent move in $A \Rightarrow C$. In that case, u is of the form $u_1 b u_2 a$ where a points to b. In that case $\lceil u \rceil = \lceil u_1 b \rceil a$ and since b is a player move $\lceil u_1 b \rceil a = \lceil u_1 \rceil ba$. This shows that a points in $\lceil u \rceil$.
- If the last move of $u = va$ is a non initial opponent move that is not external (i.e. not in the component $A \Rightarrow C$). Then in particular a is a player move either in X where X is either $A \Rightarrow B$ or $B \Rightarrow C$. Thus $\lceil u \rceil = \lceil v \rceil a$. By induction the player view $\lceil v \rceil$ preserves pointers, thus we can call the previous lemma ensuring that $\lceil v_{\restriction X} \rceil$ is contained in $\lceil v \rceil$ Furthermore since we assume that $u_{\restriction X} = v_{\restriction X} a$ is P−visible thus a points into $v_{\restriction X}$. Hence since we $\lceil v_{\restriction X} \rceil$ is contained in $\lceil v \rceil$ we conclude that a points in $\lceil v \rceil$. Thus showing that $\lceil u \rceil$ conserve pointers.

□

Theorem 3.1 (Visibility is preserved under composition)**.** *Given $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ two* P*−visible strategies. Their composition $\tau \odot \sigma$ is still* P*−visible.*

Proof. Consider s a play of $\tau \odot \sigma$ and show it is P−visible. By definition there is a witness u of s belonging to $\tau \parallel \sigma$, and thus $u_{\restriction A,B}$ and $u_{\restriction B,C}$ are both P−visible, let us show that $u_{\restriction A,C} = s$ is still P−visible.

We do so by induction on u. If u is made of only one move, then that is an initial move in C. Then $u_{\restriction A,C} = u$ and is indeed player visible. Now if u is of length $n+1$ we treat the different case for the last move.

- If the last move of u is a move in B then $u = u'a$ and $u_{\restriction A,C} = u'_{\restriction A,C}$ which is player visible by induction hypothesis.
- If the last move is an initial opponent move in $A \Rightarrow C$, then $u = u'a$ and u' satisfies the visibility condition by induction, but so does a since it has no pointer, so we conclude.
- If the last move is a non initial move in $A \Rightarrow C$ then $u = ta$ and we know a points in $\lceil ta \rceil = \lceil u \rceil$ calling the previous lemma. Thus it follows that a points in $\lceil u_{\restriction A,C} \rceil$ in $u_{\restriction A,C}$. And so $u_{\restriction A,C}$ is P−visible.

□

3.4 Well−Bracketed strategies and their composition

Definition 3.4 (Pending question, well−bracketed play)**.** Given an arena A and s a legal play on A. We say that s_i *answers to* s_j if s_i points to s_j in s, while s_i is an answer and s_j is a question. A question s_j in s is *answered* if there exists a move s_i in s that answers to s_j. Ohterwise we say that the question is *unanswered* A question is *pending* in s if it is the last unanswered question in s. We say that a play is *well−bracketed* if for each answers a in s the move a answers to the pending question in $s_{<a}$.

A strategy $\sigma : A$ is *well−bracketed* if for all its plays s the player view $\lceil s \rceil$ is well−bracketed.

Lemma 3.3 (Pending question invariance under views)**.** *Given an arena A and s a well−bracketed play on A. If the player* P *is to move in s then the pending question of s and in $\lceil s \rceil$ are the same. If the opponent is to move in s then the pending question in s and in $\lfloor s \rfloor$ are the same.*

Proof. Consider s a well bracketed play on A. We reason by induction on s. If $s = \varepsilon$ is empty then the implication is indeed true as the player is not to move. Also there is no pending question in ε and $\lfloor \varepsilon \rfloor$ and $\lceil \varepsilon \rceil$ which allows us to conclude. For the induction consider a play of the form $s \cdot m$. We treat first the case where the player is to play:

- if the player is to move and if m is an initial move. Then m must be a question and so it is the pending question of $s \cdot m$. Furthermore by definition $\lceil s \cdot m \rceil = m$, and m is obviously the pending question of m.
- If the player is to move and if m is not initial. In that case the sequence is of the form $s_1 n s_2 m$ where n justifies m. In that case $\lceil s \cdot m \rceil = \lceil s_1 \rceil nm$. Let us treat two cases:

- If m is a question: then it is the pending question of both $s \cdot m$ and $\lceil s_1 \rceil nm$ since its the last move of these sequences, so we can conclude.
- If m is an answer: then since n enables m, this means m answers n. Hence *since s is well–bracketed* the pending question in s must be n thus the pending question in $s \cdot m$ and s_1 is the same.
  Applying the induction hypothesis on s_1, we deduce that the pending question in s_1 and $\lceil s_1 \rceil$ are the same (and so it also the one of $s \cdot m$). Furthermore since m answers n the plays $\lceil s_1 \rceil$ and $\lceil s_1 \rceil nm$ have the same pending question, which allows us to conclude that $s \cdot m$ and $\lceil s_1 \rceil nm = \lceil s \cdot m \rceil$ have the same pending question.

- If the opponent is to move and there is no pending question in $s \cdot m$. Then $s \cdot m = s_1 n s_2 m$ where n justifies m and m is an answer. By the bracketing condition, there can be no pending question in s_1. Applying the inductive hypothesis on s_1 there is no pending question in both s_1 and $\lfloor s_1 \rfloor$. Thus there is no pending question in $\lfloor s_1 \rfloor nm$ which corresponds to $\lfloor sm \rfloor$, and so we conclude.

Now let us treat the case where the opponent is to play:

- If the opponent is to move and there is a pending question in sm, if m is a question then it is the pending question of sm if m is not justified then $\lfloor sm \rfloor = \lfloor s \rfloor m$ and so m is indeed the pending question of m. If m is justified and $sm = s_1 n s_2 m$ where n justifies m then $\lfloor sm \rfloor = \lfloor s_1 \rfloor nm$ of which m is also the pending question.
  If on the other hand m is an answer then it must be justified by some question n where $sm = s_1 n s_2 m$. By the bracketing condition the pending question of sm must be the pending question of s_1 as no unanswered question can occur in between n and m. Applying the inductive hypothesis on s_1 we ensure that the pending question in s_1 and $\lfloor s_1 \rfloor$ are the same and so that it is the same in $\lfloor s_1 \rfloor sm = \lfloor sm \rfloor$ this allows us to conclude.

$\square$

Lemma 3.4. *Given $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ two well bracketed strategies. For any $u \in I(A, B, C)$ if $\lceil u \rceil$ (in $A \Rightarrow C$) is in $\sigma \parallel \tau$ then $\lceil u \rceil$ is well bracketed.*

Proof. We reason by induction on $\lceil u \rceil$. If $\lceil u \rceil$ is empty then it is indeed well–bracketed. For the induction consider a $\lceil u \rceil$ to be of the form va.

- If a is a question then since by induction v is well bracketed so is va.
- If a is an answer it is a response in the component $A \Rightarrow B$ or $B \Rightarrow C$ and thus of σ or τ. Assume it is a response of σ since σ is well bracketed, $\lceil va_{\upharpoonright A,B} \rceil$ is well bracketed. Hence a answers to the pending question in $\lceil v_{\upharpoonright A,B} \rceil$. Calling the previous lemma we know that $\lceil v_{\upharpoonright A,B} \rceil$ and $v_{\upharpoonright A,B}$ have the same pending question, hence a answers to the pending question of $v_{\upharpoonright A,B}$.
  If a is a move in the arena A, since σ was to play in v the pending question in v and $v_{\upharpoonright A,B}$ must have been the same, since a move in A cannot answer to a move in C. Thus we conclude, since – by induction – v is well bracketed and a answers the pending question in v, then va is also well bracketed.
  If on the other hand, the move a is in B then the play va can be decomposed in $u_1 q u_2 a$ where a answers to q. Necessarily q is a move of τ in the arena B. Thus calling the induction hypothesis the pending question of u_1 is in $B \Rightarrow C$, but the pending question of u_1 and of $u_1 q u_2 a = va$ is the same, thus τ must play.
- If a is an answer from the external opponent i.e. on the component $A \Rightarrow C$, since va is a view it necessarily points onto its preceding move. Thus the last move of v is a question and by induction v is well bracketed, since a answers to the pending question of v we conclude that va is well bracketed.
  More precisely the play is of the form wqa where a answers q and q was a move of σ (resp. τ). The pending question of wqa is the pending question of w but by induction hypothesis this question was in $A \Rightarrow B$ (resp. $B \Rightarrow C$). Since a is a move on $A \Rightarrow C$ it is still σ (resp. τ) that must play.

□

Theorem 3.2 (Stability of Well–bracketing)**.** *Let $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ be well–bracketed strategies, then their composition $\tau \odot \sigma$ is still well–bracketed.*

Proof. Consider u a view of $\sigma \parallel \tau$ thanks to the previous lemma u is well bracketed. If now we restrict u to A and C we only remove pairs of question/answers, thus $u_{\upharpoonright A,C}$ is well bracketed. Now for any s in the composition $\tau \odot \sigma$ there is a witness u in $\tau \parallel \sigma$ such that $u_{\upharpoonright A,C} = s$ and $\ulcorner s \urcorner = \ulcorner u_{\upharpoonright A,C} \urcorner = u_{\upharpoonright A,C}$ which is well bracketed. □

3.5 Innocence

Definition 3.5 (Trace of player views)**.** Given $\sigma : A$ a strategy we define *its set of* P*–views* that we denote $\ulcorner \sigma \urcorner$ as follow $\ulcorner \sigma \urcorner = \{\ulcorner s \urcorner \mid s \in \sigma\}$. Given V a set of P–views a *trace* of V is any play s such that for all the prefix $s' \sqsubseteq^{\mathsf{P}} s$ we have $\ulcorner s' \urcorner \in V$. We denote the set of the traces of V by TrV

Proposition 3.3 (Correspondance)**.** *The pair of functions $(Tr\cdot, \ulcorner \cdot \urcorner)$ forms a Gallois correspondance between the spaces $(\mathcal{V}, \subset)$ and $(\mathcal{S}trat, \subset)$.*

Meaning $\sigma \subset TrV \Leftrightarrow \ulcorner \sigma \urcorner \subset V$, for a strategy σ and a set of player views V.

Proof. Consider $\sigma : A$ a strategy and V a set of P–views. Assume that $\ulcorner \sigma \urcorner \subset V$ consider $s \in \sigma$ and a prefix $s' \sqsubseteq^{\mathsf{P}} s$. Necessarily since $sigma$ is prefix closed $s' \in \sigma$ and thus $\ulcorner s' \urcorner \in \ulcorner \sigma \urcorner$. Thus using the hypothesis $\ulcorner s' \urcorner \in V$, this means s is a trace of V hence $s \in TrV$. So we have shown $\sigma \subset TrV$. On the other hand assume $\sigma \subset TrV$ consider $v \in \ulcorner \sigma \urcorner$ thus there exists s a play of σ such that $\ulcorner s \urcorner = v$, but by hypothesis $s \in TrV$ and since in particular s is a prefix of itself $\ulcorner s \urcorner = v$ is in V. □

Definition 3.6. The *saturation* of a strategy σ is defined as $\overline{\sigma} = Tr\ulcorner s \urcorner$. We say that a strategy σ is *saturated* if $\sigma = \overline{\sigma}$

Definition 3.7 (Innocent strategy)**.** A strategy $\sigma : A$ is *innocent* if it is P–visible and:

$$\text{for any } sa^+ \in \sigma \text{ and } t \in \sigma \text{ if } \ulcorner s \urcorner = \ulcorner t \urcorner \text{ then } ta^+ \in \sigma \text{ and } \ulcorner sa^+ \urcorner = \ulcorner ta^+ \urcorner.$$

Proposition 3.4 (Characterization of Innocence)**.** *Given a strategy $\sigma : A$ it is equivalent that:*

- *σ is innocent.*
- *σ is visible and saturated.*

Proof. Consider that σ is innocent. Let us show σ is visible, consider s a play in σ and show $\ulcorner s \urcorner$ is still a legal play. We reason by induction on the size of s, if s is empty then the statement is obvious. Now consider a play of the form sa let us treat cases based on the nature of the move a.

- If a is an initial move then $\ulcorner sa \urcorner = a$ and thus the play is visible.
- If a is not initial and is an opponent move then sa is of the form s_1bs_2a where a points to b, and $\ulcorner sa \urcorner = \ulcorner s_1b \urcorner a$. By induction $\ulcorner s_1b \urcorner$ is a legal play; furthermore a is an opponent move and thus b is a player move now since $\ulcorner s_1b \urcorner$ and s_1b have the same finishing move that is b we conclude that $\ulcorner s_1b \urcorner$ is a legal play ending by a player move. Thus $\ulcorner s_1b \urcorner a$ is a legal play, meaning that $\ulcorner s \urcorner$ is.
- If a is not initial and is a player move. Then we have $\ulcorner sa \urcorner = \ulcorner s \urcorner a$ by induction $\ulcorner s \urcorner$ is a legal play.

Let us show that σ is saturated. Consider s

□

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